**Dynamics of the Mantle and Lithosphere – ETH Zürich**

# Continuum Mechanics in Geodynamics: Equation cheat sheet

**(or all equations you will probably ever need)**

## Definitions

1. Coordinate system. $(x, y, z)$ or $(x_1, x_2, x_3)$ define points in 3D space.
2. Field (variable). $T(x, y, z)$ - temperature field - temperature varying in space
3. Indexed variables. $v_i, i = 1, 2, 3$ implies $(v_1, v_2, v_3)$, i.e. three functions.
4. Repeated indices rule (also called Einstein summation convention)

$$\frac{\partial v_i}{\partial x_i}, i = 1,2,3, implies \sum_{i=1}^{3} \frac{\partial v_i}{\partial x_i} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}$$

5. Tensor = indexed variable + the rule of transformation to another coordinate system.
6. Useful tensor - Kronecker delta, in 3D:

$$\delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

7. Traction = a force per unit area acting on a plane (a vector).
8. Traction sign convention. Compression is negative in mechanics, but positive in geology.
9. Mean stress/strain: $\bar{\sigma} = \sigma_{ii}/3 = tr(\sigma)/3, \bar{\varepsilon} = \varepsilon_{ii}/3 = \theta$ (also called dilatation)
10. Deviatoric stress/strain: $\tilde{\sigma} = \sigma_{ij} - \bar{\sigma}\delta_{ij}, \tilde{\varepsilon} = \varepsilon_{ij} - \bar{\varepsilon}\delta_{ij}$

## Stress tensor

1. What is it?
   a matrix, two indexed variables, tensor of rank two:

$$\sigma_{ij} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} (2D)$$

$$\sigma_{ij} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} (3D)$$

2. Meaning of the elements:
   each row are components of the traction vectors acting on the coordinate planes.
   diagonal elements are normal stresses.
   off diagonal elements are shear stresses.
3. Special properties: Symmetric for homogeneous material.

4. What for?
   It is a magic tool: if you multiply the stress tensor (treated as a simple matrix) by a unit vector, $n_j$, which is normal to a certain plane, you will get the traction vector on this plane (Cauchy's formula):

$$T_{(n)i} = \sigma_{ij} n_j = \sum_{j=1}^{3} \sigma_{ij} n_j$$

5. How do you get it?
   Usually by solving the equilibrium equations

6. NB: The number of equilibrium equations is less than the number of unknown stress tensor components.

## Strain and strain rate tensors

1. What is it?
   a matrix, two indexed variables, tensor of rank two.

2. Meaning of the elements:
   diagonal elements are elongation (relative changes of length in coordinate axes directions).
   off diagonal elements are shears (deviations from 90° of the angles between lines coinciding with the coordinate axes directions before deformation).

3. Special properties: symmetric

4. What for?
   It is a measure of the deformation. It will be used in the rheological relationships.

5. How do you get it?
   via velocity/displacements:

$$\begin{aligned} \dot{\varepsilon}_{ij} &= \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \\ &= \begin{pmatrix} \frac{\partial v_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2}\right) & \frac{\partial v_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial v_3}{\partial x_1} + \frac{\partial v_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial v_3}{\partial x_2} + \frac{\partial v_2}{\partial x_3}\right) & \frac{\partial v_3}{\partial x_3} \end{pmatrix} \end{aligned}$$

6. NB: The number of velocity components is smaller than the number of strain rate components.

## Rheology, Stress-strain relationships

1. What is it?
   A functional relationship between second rank tensors:

   *incompressible viscous:* $\sigma_{ij} = -P\delta_{ij} + 2\eta\dot{\varepsilon}_{ij}$

   *elastic:* $\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$

   *Maxwell visco-elastic (for deviators):* $\dot{\varepsilon}_{ij} = \frac{\dot{\tilde{\sigma}}_{ij}}{2G} + \frac{\tilde{\sigma}_{ij}}{2\eta}$

2. Meaning of the elements:
   $\lambda, \eta, \mu$ - are parameters (material properties)

3. What for?
   In the equilibrium equations: first substitute stress via strain(rate), than strain(rate) via displacement (velocities), which results in a "closed" system of equations, meaning that the number of equations is equal to the number of unknowns (velocities or displacement).

4. How to find?
   measure rheology in the lab.

5. NB. There are three major classes of rheologies:

   - Reversible elastic rheology at small stresses and strains.
   - Rate dependent creep - irreversible. Examples are Newtonian viscous or power-law rheology, which is usually thermally activated, pressure independent. Intermediate stress levels.
   - Rate independent (instantaneous) ultimate yielding at large stresses. Frequently pressure sensitive, temperature independent. Also called: plastic or frictional (brittle) behavior.

## General continuum mechanics recipe: How to derive a closed system of equations.

1. Conservation laws
   Conservation of mass:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_i}{\partial x_i} = 0$$

   Conservation of momentum ("equilibrium" force balance)

$$\rho \left( \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} \right) = \frac{\partial \sigma_{ij}}{\partial x_j} + \rho g_i$$

   where $g_i$ is the gravitational acceleration vector.

   Conservation of energy:

$$\left( \frac{\partial E}{\partial t} + v_j \frac{\partial E}{\partial x_j} \right) + \frac{\partial q_i}{\partial x_i} = \rho Q$$

   where $E$ is energy, $q_i$ the energy flux vector, and $Q$ an energy source (heat production).

2. Thermodynamic relationships

   Equations of state 1: the "caloric" equation

$$E = c_p \rho T$$

   where $c_p$ is heat capacity, and $T$ is temperature.

   Equations of state 2: relationships for the isotropic parts of the stress/strain tensors

$$\rho = f(T, P)$$

   where $P$ is pressure (note: $\rho = \rho_0 \varepsilon_{kk}; P = -\bar{\sigma} = -\sigma_{kk}/3$).

3. Rheological relationships for deviators

$$\tilde{\dot{\varepsilon}}_{ij} = R(\tilde{\dot{\sigma}}_{ij}, \tilde{\sigma}_{ij})$$

4. Energy flux vector vs. temperature gradient

$$q_i = -k\frac{\partial T}{\partial x_i}$$

where $k$ is the thermal conductivity.

## Summary: The general system of equations for a continuum media in the gravity field.

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_i}{\partial x_i} &= 0 && (1)\\
\rho\left(\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{x_j}\right) &= \frac{\partial \sigma_{ij}}{\partial x_j} + \rho g_i && (2)\\
\left(\frac{\partial E}{\partial t} + v_j\frac{\partial E}{x_j}\right) + \frac{\partial q_i}{\partial x_i} &= \rho Q && (3)\\
E &= c_p \rho T && (4)\\
\rho &= f(T,P) && (5)\\
\tilde{\dot{\varepsilon}}_{ij} &= R(\tilde{\dot{\sigma}}_{ij}, \tilde{\sigma}_{ij}) && (6)\\
q_i &= -k\frac{\partial T}{\partial x_i} && (7)
\end{aligned}$$

where $\rho$ is density, $v_i$ velocity, $g_i$ gravitational acceleration vector, $E$ energy, $q_i$ heat flux vector, $Q$ an energy source (heat production, *e.g.* by radioactive elements), $c$ is heat capacity, $T$ temperature, $P$ pressure and $k$ thermal conductivity.
Known functions, tensors and coefficients: $g_i, c_p, f(..), \rho_0, R(...), k$
Unknown functions: $\rho, v_i, \bar{\sigma}, \tilde{\sigma}_{ij}, q_i, E, T$. The number of unknowns is thus equal to the number of equations.

**Example: The Stokes system of equations for slowly moving incompressible linear viscous (Newtonian) continuum materials.**

$$\frac{\partial v_i}{\partial x_i} = 0 \quad (8)$$

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho_0 g_i = 0 \quad (9)$$

$$\rho_0 c_p\left(\frac{\partial T}{\partial t} + v_j\frac{\partial T}{x_j}\right) = \frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i}\right) + \rho_0 Q \quad (10)$$

$$\tilde{\dot{\varepsilon}}_{ij} = \frac{\tilde{\sigma}_{ij}}{2\eta} \quad (11)$$

$$\sigma_{ij} = -P\delta_{ij} + \tilde{\sigma}_{ij} \quad (12)$$

Known functions, tensors and coefficients: $g_i, c_p, Q, \rho_0, \eta, k$
Unknown functions: $v_i, P, \tilde{\sigma}_{ij}, \sigma_{ij}, T$. The number of unknowns is thus equal to the number of equations.

**2D version, spelled out**

Choice of coordinate system and new notation for 2D:
$g_i = (0, -g), x_i = (x, z), v_i = (v_x, v_z), \sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{zx} & \sigma_{zz} \end{pmatrix}$ Note that $\sigma_{zx} = \sigma_{xz}$.
The 2D Stokes system of equations (the basis for basically every mantle convection/lithospheric deformation code):

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z} = 0 \quad (13)$$

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} = 0 \quad (14)$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} - \rho g = 0 \quad (15)$$

$$\sigma_{xx} = -P + 2\eta \frac{\partial v_x}{\partial x} \quad (16)$$

$$\sigma_{zz} = -P + 2\eta \frac{\partial v_z}{\partial z} \quad (17)$$

$$\sigma_{xz} = \eta \left( \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right) \quad (18)$$

$$\rho_0 c_p \left( \frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_z \frac{\partial T}{\partial z} \right) = k \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} \right) + \rho_0 Q \quad (19)$$

Known: $g, Q, c_p, \rho_0, \eta, k$. Unknown: $v_x, v_z, P, \sigma_{xx}, \sigma_{xz}, \sigma_{zz}, T$. Number of equations?