Chapter 4

Fluid-dynamical equations and convection

The mantle is believed to sub-solidus convection. The hot material at depth and cold material near the Earth's surface provides a gravitationally instable situation: the hot material wants to rise, and the cold material wants to sink. If the strength of mantle material allows for the necessary deformation to accomplish this, very slow convection will occur (velocities in the order of a few cm/yr), in which mantle material will behave like a fluid. The surface expression of convection is plate tectonics, and the presence of plate tectonics, together with the seismic images of the Earth's interior temperature field, provides direct evidence of convection underneath the moving plates.

4.1 Governing equations

On large timescales, the mantle can therefore be regarded as a (very viscous) fluid, and we can use a general fluid-dynamical approach to describe the flow of mantle material. We use conservation of mass (the continuity equation) and momentum (the momentum or Stokes equation) to uniquely describe the flow field. Since convective flow is driven by internal density variations, an equation to describe the density structure (an equation of state) is required. The constitutive equation, discussed in some detail in the former section, describes the deformation behaviour under applied stresses. Finally, since both rheology and density both heavily depend on the thermal structure, the temperature distribution is solved by applying energy conservation. Here, we will elaborate each of these equations, and non-dimensionalize them, before we continue with some applications and analytical solutions.

4.1.1 Continuity equation

The continuity equation says that mass is conserved. This implies that within a fixed volume, or 'control volume', a change in density must be accompanied by in- or outflow of material. For most applications, the mantle is regarded as an incompressible fluid. Using the Boussinesq approximations, to be discussed later, this allows us to take the density of the fluid as a constant in the continuity equation. This implies that the net in- and outflow of material into the control volume is zero. This is illustrated for two dimensions in Figure 4.1. The net flow rate out of an infinitesimal rectangular element in x-direction is $\frac{\partial u}{\partial x}\delta x$. Multiplied by the cross-sectional area δy, this gives the net amount of material flowing out of the element in x-direction. Similarly, the net amount of material flowing out of the element in y-direction is: $\frac{\partial v}{\partial y}\delta y \times \delta x$. Adding and dividing by the volume $\delta x \delta y$ of the element gives the net total outflow per unit area $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$. Conservation of mass in an incompressible fluid requires that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{4.1}$$

More general, for one, two, or three dimensions, this is rewritten as:

$$\nabla \cdot \vec{u} = 0 \tag{4.2}$$

Or (in the Einstein notation)

$$\frac{\partial v_i}{\partial x_i} = 0 \tag{4.3}$$

where repeated indices indicate summation.

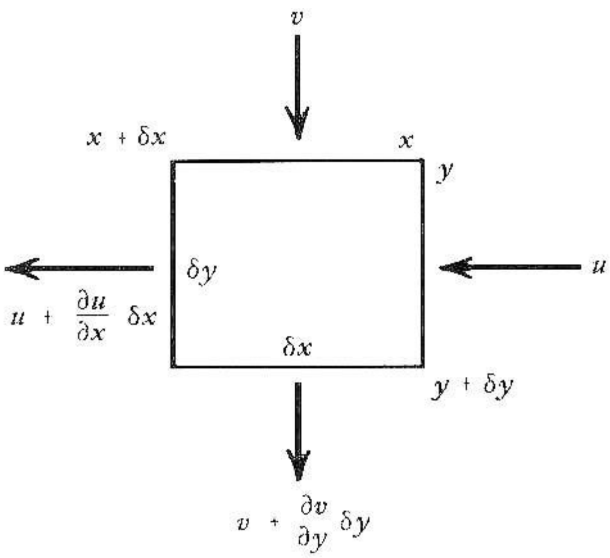

Figure 4.1: Mass conservation in 2-D of an infinitesimal rectangular element. From [Turcotte and Schubert, 2002, Fig.6-10]

4.1.2 Momentum equation

Because the mantle has such a high viscosity, inertial forces can be neglected. We can apply a force balance of the pressure forces, dynamic stresses, and gravity forces separately on the same infinitesimal rectangular element as used for constructing the mass conservation equation. Figure 4.2 illustrates this for the two-dimensional case. Starting with the pressure forces: adding all forces and dividing by the volume of the element gives:

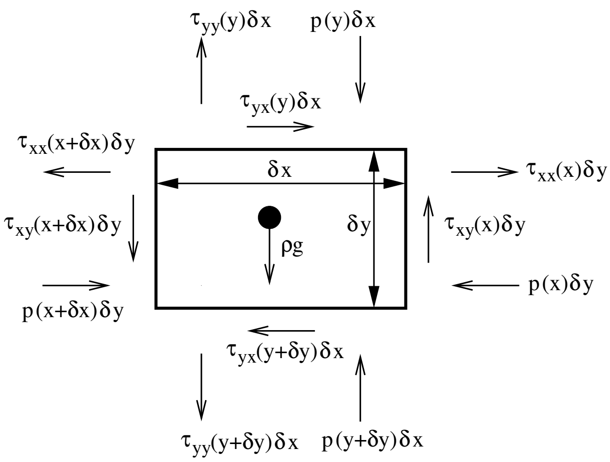

Figure 4.2: Conservation of momentum in 2-D of an infinitesimal rectangular element

$$\frac{p(x+\delta x)-p(x)}{\delta x}+\frac{p(y+\delta y)-p(y)}{\delta y}=\frac{\partial p}{\partial x}+\frac{\partial p}{\partial y} \tag{4.4}$$

using a Taylor expansion. We can derive similar expressions for the deviatoric normal stresses τ_{xx} and τ_{yy}

$$\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y} \tag{4.5}$$

and the deviatoric shear stresses τ_{xy} and τ_{yx}

$$\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y} \tag{4.6}$$

τ_{xy} and τ_{yx} are the same to avoid a net torque on the infinitesimal element. Using the definition of (effective) viscosity, we rewrite the stress terms τ_{ij}:

$$\tau_{xx} = 2\eta\frac{\partial u}{\partial x} \tag{4.7}$$

$$\tau_{yy} = 2\eta\frac{\partial v}{\partial y} \tag{4.8}$$

$$\tau_{xy} = \tau_{yx} = \eta\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{4.9}$$

Adding all the terms that act in the x-direction and using mass conservation Equation 4.1 to replace the resulting

$$\frac{\partial^2 v}{\partial x\partial y} = \frac{\partial^2 u}{\partial x^2} \tag{4.10}$$

gives the momentum equation in x-direction:

$$0 = -\frac{\partial p}{\partial x} + \eta\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{4.11}$$

where the minus-sign in front of the pressure term arises from the unfortunate convention that pressure is positive inwards (compressional), while deviatoric normal stress is positive outwards (tensional). A similar derivation of the momentum equation in y-direction can be obtained, and after adding the gravity body force ρg (which acts only in the vertical direction, i.e. the y-direction in this case)

$$0 = -\frac{\partial p}{\partial y} + \eta\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \rho g \tag{4.12}$$

At large depth the lithostatic pressure becomes large in comparison to the deviatoric stresses. Therefore, this pressure is usually subtracted by defining the density and pressure deviation from the lithostatic one: $\Delta\rho = \rho - \rho_0$, and $\Delta p = p - \rho_0 g y$. Now the momentum equations become:

$$0 = -\frac{\partial \Delta p}{\partial x} + \eta\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{4.13}$$

$$0 = -\frac{\partial \Delta p}{\partial y} + \eta\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \Delta\rho g \tag{4.14}$$

We can write the momentum equation more generally for one, two or three dimensions, and for variable viscosity $\eta = \eta(x_i)$:

$$0 = -\frac{\partial \Delta p}{\partial x_i} + \frac{\partial(\eta\dot{\varepsilon}_{ij})}{\partial x_j} + \Delta\rho g \delta_{i3} \tag{4.15}$$

The momentum equation with negligible inertia terms is also referred to as the Stokes' equation.

Exercise: Show that Equation 4.15 simplifies to Equation 4.13 or 4.14 by taking $i = x$ or y, and η is constant.

4.1.3 Equation of state

Flow in the Earth is driven by local variations in the density, caused by mainly temperature and compositional differences. Assuming incompressibility, and the presence of only one material, the equation of state, which tells us something how temperature and pressure are related to density, reduces to:

$$\rho = \rho_0\left(1 - \alpha(T - T_0)\right) \tag{4.16}$$

where α is the thermal expansion coeeficient as discussed in the first part of this course.

4.1.4 Constitutive equation

The constitutive equation gives the deformation of a material as a function of the ambient temperature, stress, etc. It is discussed in the former Chapter. A general non-Newtonian Arrhenius type flow law looks like:

$$\dot{\varepsilon}_{ij} = A d^m \sigma^n \exp(\frac{E + pV}{RT}) \tag{4.17}$$

with d the grainsize, σ the deviatoric stress, E the activation energy, and V the activation volume. Diffusion creep has $n = 1$, and $m \approx 3$, and dislocation creep has $m = 0$ and $n \approx 3$.

4.1.5 Energy equation

Both density and rheology are strongly dependent on the ambient temperature, and it is therefore necessary to accurately determine the temperature distribution in the Earth, by solving the energy equation. This equation is discussed in the first part of the course. Here, we neglect effects on the temperature by adiabatic (de)compression and viscous heating, and radiogenic heat production. In general, these are all important in the Earth, but for the moment, we are mainly interested in the temperature effects on the flow field:

$$\rho C_p \frac{\partial T}{\partial t} = \nabla \cdot k \nabla T - \rho C_p \vec{v} \cdot \nabla T \tag{4.18}$$

4.1.6 Boussinesq approximations

As mentioned before, the density differences in the Earth drive thermal convection, and are responsible for most of the Earth's dynamics. So in the momentum equation 4.15, the $\Delta\rho g$ term is essential. But in most other respects, however, the variations in the density appearing in the governing equations can be neglected. Density variations in the continuity equation were already neglected in deriving Equation 4.2. If we furthermore adopt the simplified energy equation 4.18 by neglecting adiabatic compression, viscous heating and radiogenic heat production, we have a set of simplifying assumptions which are called the Boussinesq approximations. These approximations are widely used in geodynamics and other applications of fluid dynamics. But we need to keep in mind that those assumptions are not always valid. For instance, when dealing with the longterm evolution of the Earth's thermal regime, we have to take radiogenic heat production into account. When dealing with rapid up- and downwellings in the mantle, such as subduction zones, and mantle plumes, adiabatic (de)compression can sometimes be important. As a final example, frictional or viscous heating can be significant in narrow shear bands.

4.2 Streamfunction formulation

In an incompressible 2-D flow, the two components of the velocity field are related through the continuity equation 4.2. We can implicitly satisfy the continuity equation by introducing the *streamfunction* Ψ:

$$u = -\frac{\partial \Psi}{\partial y} \tag{4.19}$$

$$v = \frac{\partial \Psi}{\partial x} \tag{4.20}$$

The velocity components u and v are then usually referred to as primitive variables.

Exercise: Show that Equation 4.2 is always implicitly satisfied when using the stream function formulation.

With this streamfunction formulation, the Stokes' equation can be written as

$$\nabla^4 \Psi = \eta^{-1} \frac{\partial \Delta\rho g}{\partial x} \tag{4.21}$$

which is known as the inhomogeneous biharmonic equation. In the absence of gravity, the right-hand-side would be zero, which would make the biharmonic equation homogeneous.

The streamfunction is an intergral of the velocity field over a distance perpendicular to the flow direction ($\Psi = \int_x u_y dx$ or $\Psi = \int_y u_x dy$). Therefore, the streamfunction value can be physically interpreted as the fluid flux, or volumetric rate of flow between two points. The absolute value of the streamfunction is rather meaningless: it is the difference in streamfunction value that has a physical meaning.

4.3 Example: mantle viscosity determination from post-glacial rebound

Now that the governing equations are discussed, and the streamfunction formulation is introduced, we can apply the Stokes' equation to determine the mantle viscosity from the post-glacial rebound data. Major ice sheets were covering continental areas of, for example Canada and Fennoscandia during the last ice age. Those ice sheets depressed the surface. By doing so, viscous mantle material below the lithosphere was pushed away. After the ice age disappeared, the reverse occurred, which is a process that still continues today: gravity forces make mantle material flowing back into the 'gap' left behind by the ice sheet (Figure 4.3). The speed at which this occurs is

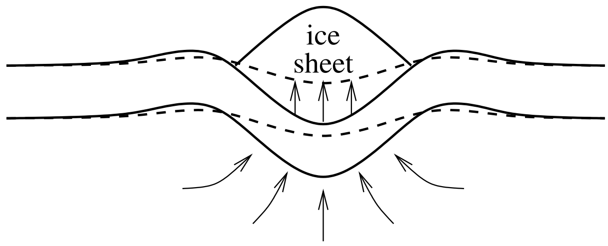

Figure 4.3: Flow of mantle material during post-glacial rebound

dependent on the viscosity of the mantle material. We will now see how the mantle viscosity can be determined. The original analysis presented here was given by [Haskell, 1935], and the elaboration given here closely follows the ones given by Turcotte and Schubert [2002] and Ranalli [1995]. Suppose the depression h of the lithosphere at the time of deglaciation (which we will call $t = 0$ here) is given by

$$h(t = 0) = h_0 \cos 2\pi x/\lambda \tag{4.22}$$

with the wavelength $\lambda \gg h_0$. It is reasonable to assume that the velocity field will also be a harmonic function with the same wavelength, and therefore also the streamfunction. It will turn out that after applying separation of variables

$$\Psi = sin\frac{2\pi x}{\lambda} Y(y) \tag{4.23}$$

with $Y(y)$ yet to be determined. Filling in Equation 4.23 into the homogeneous version of Equation 4.21 (homogeneous because we can neglect lateral density variations in the mantle for this purpose, so the right-hand side of Equation 4.21 is zero) gives a fourth order differential equation, of which the general solution is

$$\Psi = sin\frac{2\pi x}{\lambda}\left(Ae^{-2\pi y/\lambda} + Bye^{-2\pi y/\lambda} + Ce^{2\pi y/\lambda} + Dye^{2\pi y/\lambda}\right) \tag{4.24}$$

To have a finite solution at $y \to \infty$, $C = D = 0$. Differentiating to y and x gives u and v respectively. Unknowns A and B are found by filling in the following boundary conditions at $y = 0$ (for which the velocity field is approximately the same as at the top of the lithosphere at $y = h$, if $\lambda \gg h_0$): 1) the rigid lithosphere on top of the mantle inhibits horizontal flow near the surface $u(y = 0) = 0$, and 2) the hydrostatic pressure ($\rho g h(t)$) in the 'gap' should equal the total viscous stress $\sigma = -p + 2\eta\frac{\partial v}{\partial y}$. p is then found by filling in the solution for u into the horizontal component of the Stokes equation 4.13, and integrated, and $\frac{\partial v}{\partial y}$ is found by differentiating the solution for v. The vertical velocity v is the time derivative of the vertical displacement $h(t)$. Combining all this eventually leads to the following differential equation for $h(t)$:

$$\frac{\partial h(t)}{\partial t} = -\frac{\lambda \rho g h(t)}{4\pi\eta} \tag{4.25}$$

Solving this gives

$$h(t) = h_0 \exp\left(\frac{-\lambda\rho g t}{4\pi\eta}\right) = h_0 e^{-t/\tau_r} \tag{4.26}$$

with τ_r the relaxation time. From sea-level observations, and free-air gravity anomalies, the relaxation time for both the Canadian and Fennoscandian rebound is estimated around 5000 years. Taking the wavelength of the 'gap' to be around 3000 km, and $\rho \approx 3300 kg/m^3$ gives an estimated mantle viscosity $\eta \approx 10^{21}$ Pa s. Note that by the principle of superposition a more complicated initial shape of the depression can be taken into account by linear superposition of several harmonic functions h_i. The relaxation time τ_r is inversely related to the wavelength λ, so that shorter wavelenths will maintain over a longer time period. More accurate and multi-layer viscosity profiles of the mantle have been obtained by taking several refinements into account, such as a visco-elastic rheology (elastic lithosphere over a viscous mantle), self-gravitation, and a more accurate deglaciation history. An example of a more accurate and detailed analysis is given in Figure 4.4.

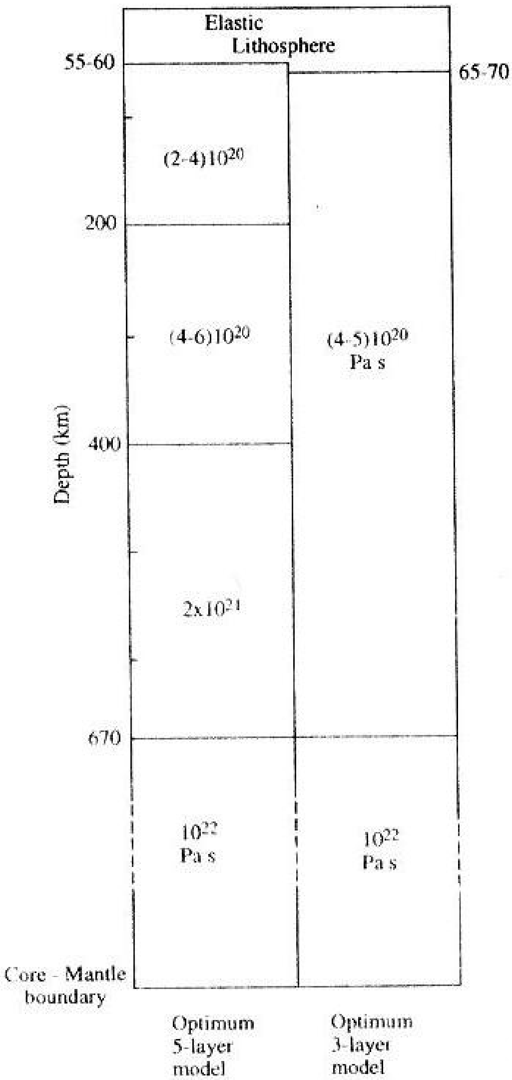

Figure 4.4: Example of the effective viscosity profile below the Brittish Isles from post-glacial rebound results. Figure from [Lambeck and Johnston, 1998, Figure 10.12]

4.4 Mantle convection

As mentioned in the beginning of this chapter, mantle convection is driven by the gravitational instable situation of a hot layer below a colder one. This can, but doesn't always, result in convection, and whether convection occurs, and with which vigor, depends on a lot of parameters: the temperature difference between top and bottom, the thermal expansivity which translates temperature into density, the size of the convection cell, the effective viscosity of the mantle material, and some more parameters. We will see that we can reduce the complexity of this system significantly if we *scale* or *non-dimensionalize* the equations.

4.5 Scaling

Scaling is almost always done in the literature as well. We scale each of the governing Equations 4.2, 4.15, and 4.18 by deviding each of the variable physical parameters by a logically chosen constant value:

$$\vec{x}' = \vec{x}/h \tag{4.27}$$

$$t' = t'/(h^2/\kappa) \tag{4.28}$$

$$T' = (T - T_0)/\Delta T \tag{4.29}$$

$$\eta' = \eta/\eta_0 \tag{4.30}$$

in which h is the height of the convection cell, $\kappa = k/(\rho C_p)$ is the thermal diffusivity, ΔT is the temperature difference between the top and bottom of the convection cell, $T_0 = T_s/\Delta T$ is the nondimensional temperature at the top of the convection cell, i.e. the surface temperature, and η_0 is the viscosity at $T' = 1$ (and given reference strainrate $\dot{\varepsilon}_0$ in case of non-Newtonian rheology. By applying the nondimensionalization, and removing primes (i.e. all quantities are from now on non-dimensional without explicit mentioning), we arrive at the following set of non-dimensional equations:

$$\nabla \cdot \vec{u} = 0 \tag{4.31}$$

$$-\nabla\Delta p + \nabla^2\vec{u} = \mathrm{R}aT\delta_{i3} \tag{4.32}$$

$$\frac{\partial T}{\partial t} + \vec{u}\cdot\nabla T = \nabla^2 T \tag{4.33}$$

with the Rayleigh number defined as: $Ra = \frac{\alpha\rho_0 g\Delta T h^3}{\eta\kappa}$.

4.5.1 Deriving $\mathrm{R}a_c$ from linear stability analysis

Suppose for simplicity that the mantle consists of a constant-viscosity fluid that is heated from below, and cooled from the top by applying constant temperatures there. For small values of the temperature difference between the top and bottom of a system (or more precisely a small Rayleigh number), heat conduction takes care of the resulting heat flow from the bottom to the top. If the temperature contrast becomes too large, the available gravitational energy is enough to overcome the frictional resistance against deformation, and the system will start to convect. With the set of non-dimensional equations and the stream function formulation, we are able to derive the critical value of Ra (called $\mathrm{R}a_c$) for which a 2-D system is marginally stable, i.e. on a transition from heat conduction to convection. First define the total temperature T as a sum of the time-independent conductive cooling solution T_0 and a deviation T_1, so that $T = T_0 + T_1$. As boundary conditions we require that T_1 goes to zero near the upper and lower boundary. We can substitute this in Equation 4.33. Since at marginal stability, both $\vec{u}$ and ∇T_1 are small, their product is neglected. This removes the nonlinear coupling between the energy and Stokes equation, and we can solve for the system analytically with sepratation of variables. Substituting the stream function formulation for $\vec{u}$ in the energy equation gives:

$$\frac{\partial T_1}{\partial t} - \nabla^2 T_1 = \frac{\partial\Psi}{\partial x} \tag{4.34}$$

while scaling Equation 4.21 gives:

$$\eta\nabla^4\Psi = Ra\frac{\partial T_1}{\partial x} \tag{4.35}$$

Without going into the details of the solution method, the general solution for this system of Equations 4.34 and 4.35 is:

$$\Psi = \Psi_0\cos(kx)\sin(n\pi z)\exp(\alpha t) \tag{4.36}$$

$$T_1 = T_{1,0}\sin(kx)\sin(n\pi z)\exp(\alpha t) \tag{4.37}$$

with k the horizontal wave length of the instability. Vertically, we require an integer number n of 'convection cells' to fit in the total system. The value of α determines the stability of the system: for $\alpha < 0$ any initial perturbation will fade away in time, while for $\alpha > 0$ it will grow and eventually lead to convection. For $\alpha = 0$, we find *marginal* stability. One can find that

$$\alpha = \frac{\mathrm{R}ak^2 - (k^2 + n^2\pi^2)^3}{(k^2 + n^2\pi^2)^2} \tag{4.38}$$

Putting it differently, the critical Rayleigh number value $\mathrm{R}a = \mathrm{R}a_c$ (for which $\alpha = 0$) is given by

$$\mathrm{R}a_c = \frac{\left(k^2 + n^2\pi^2\right)^3}{k^2} \tag{4.39}$$

A minimum $\mathrm{R}a_c$ value is found for $n = 1$ and $k = \pi/\sqrt{2}$, for which $\mathrm{R}a_c = \frac{27\pi^4}{4} = 657.48$. Turcotte and Schubert [2002] plotted critical Rayleigh number for other values of k and n, as shown in Figure 4.5.

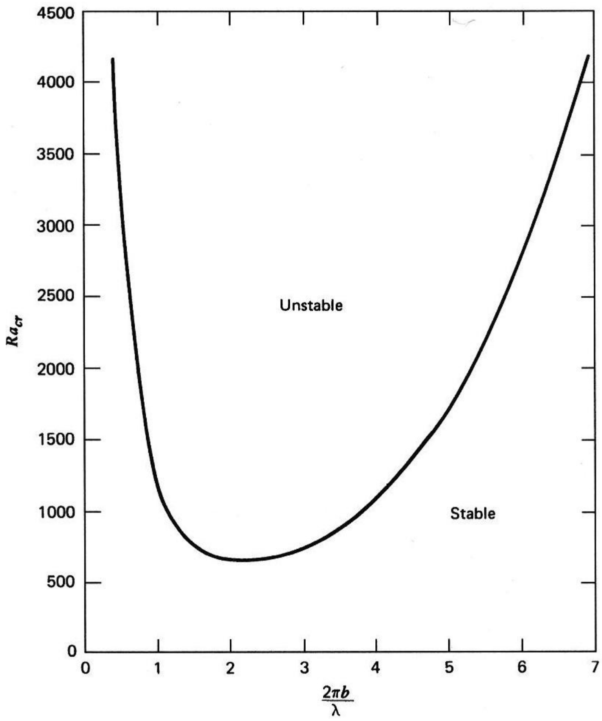

Figure 4.5: Critical Rayleigh number as a function of horizontal and vertical wavelength. Figure from Turcotte and Schubert [2002]. Their notation $\frac{2\pi b}{\lambda}$ equals $\frac{k}{n}$ in the description given here.

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