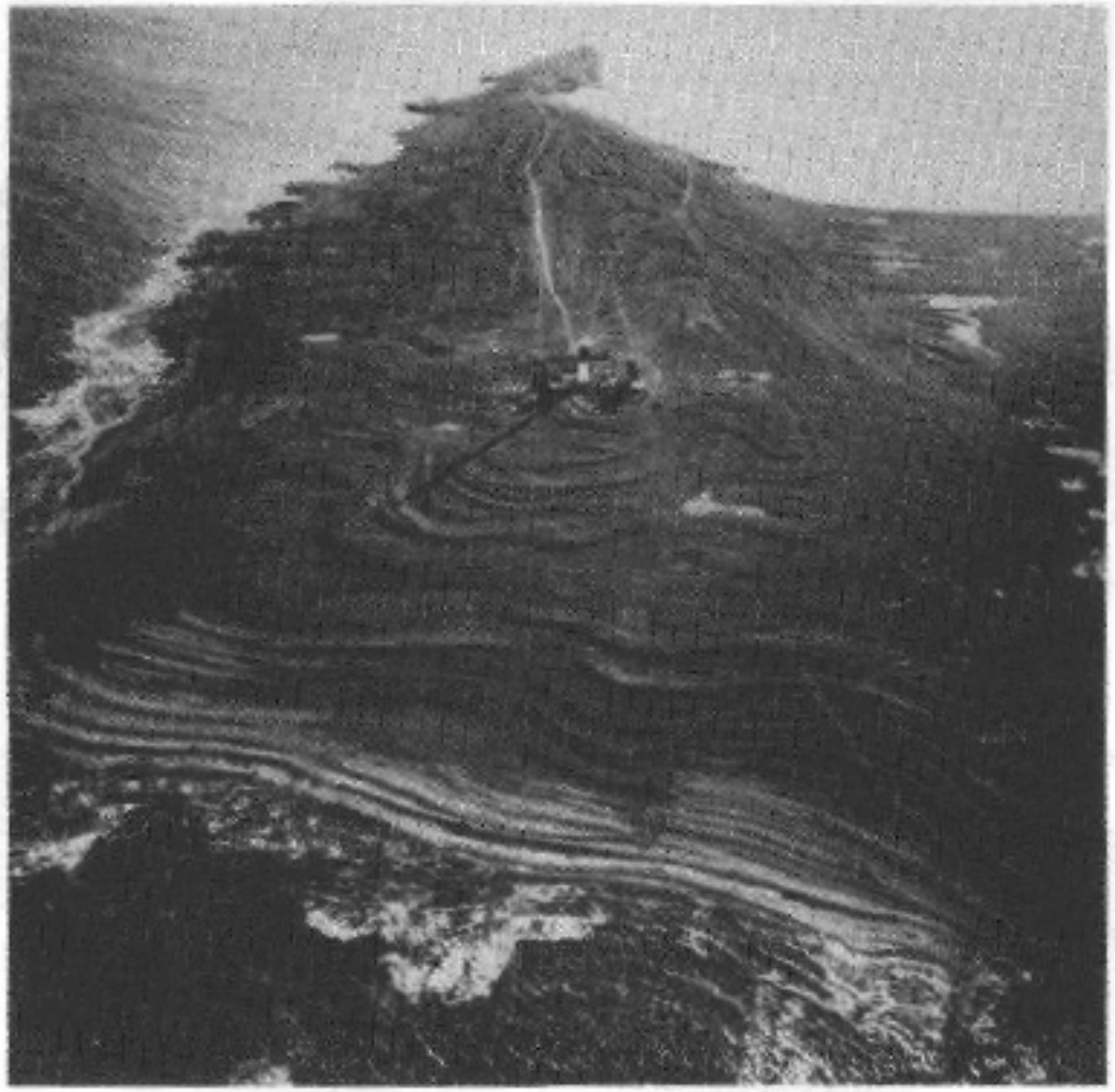
Fluid dynamics of mantle convection

(651-4008-00 G)

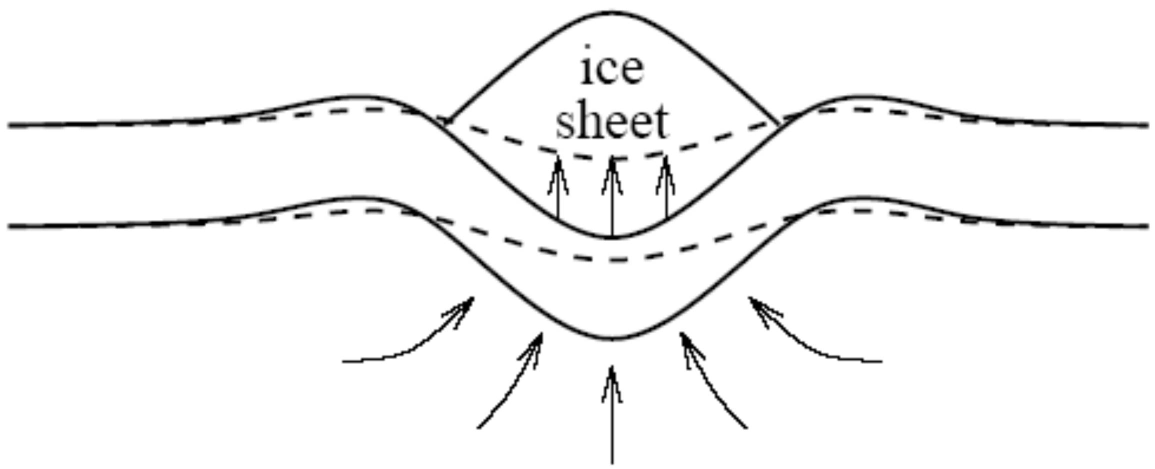
Schedule

- *Basic equations you need to know in geodynamics*
- Fundamentals of fluid dynamics
  - Vectors and tensors
  - Conservation laws and constitutive law
  - 1D examples
  - Dynamic similarity
- Fundamentals of mantle convection
  - Boussinesq approximation
  - Rayleigh number and Nusselt number
  - Stability analysis of mantle convection

Examples: Post-glacial rebound



6–15 Elevated beach terraces on Östergransholm, Eastern Gotland, Sweden. The contempory uplift rate is about 2 mm yr^{-1}. (Photographer and copyright holder, Arne Philip, Visby, Sweden; courtesy IGCP Project Ecostratigraphy.)



Initial: $h(t=0) = h_0 \cos(2\pi/Lx)$

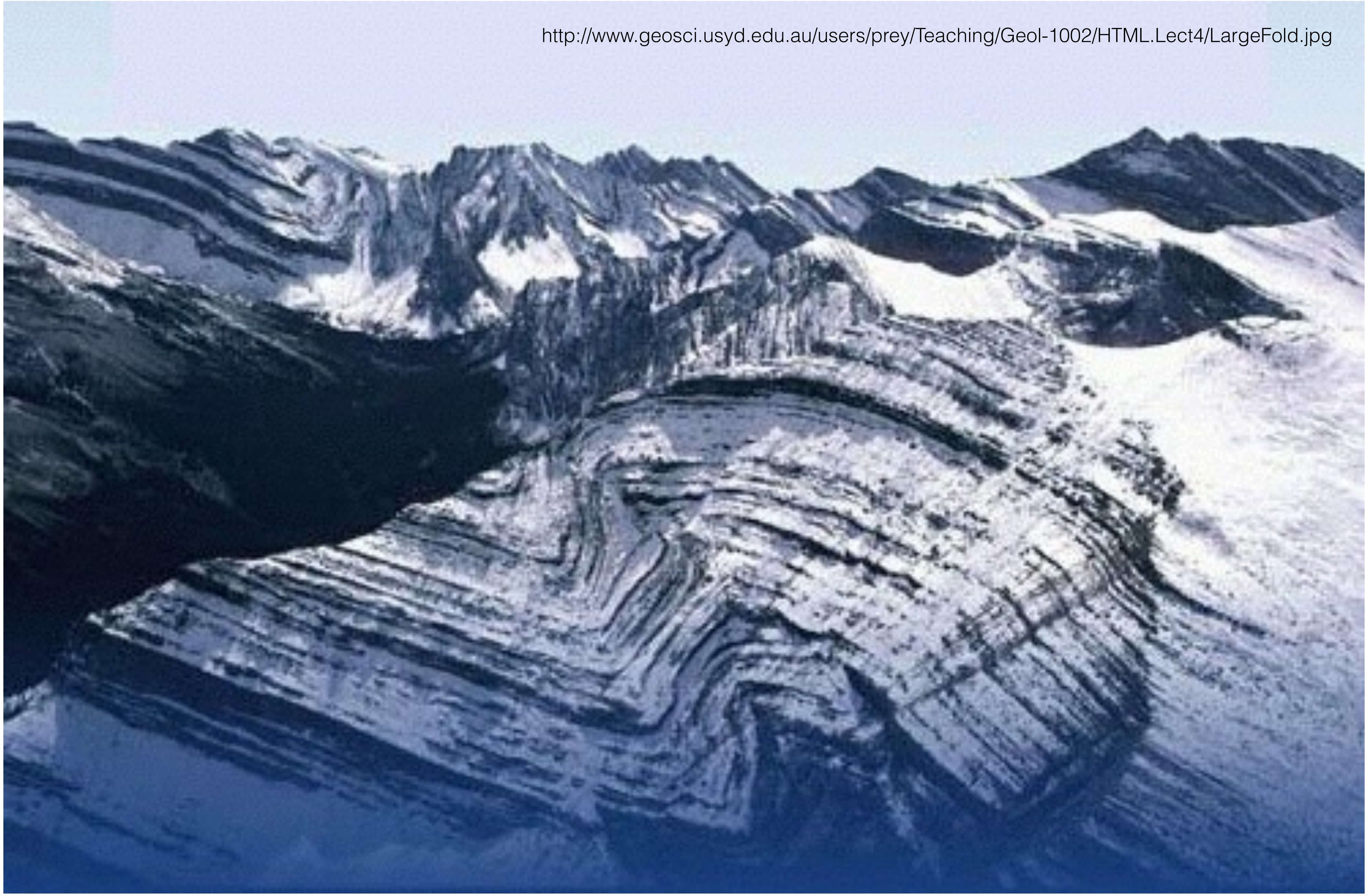
Assumption: $L >> h_0$

Coastline photo from Turcotte & Schubert, 2002)

Example: Ductile structures

***O*(1) m**

Example: Ductile structures



***O*(100) m**

Example: Ductile structures

***O*(1000) km**

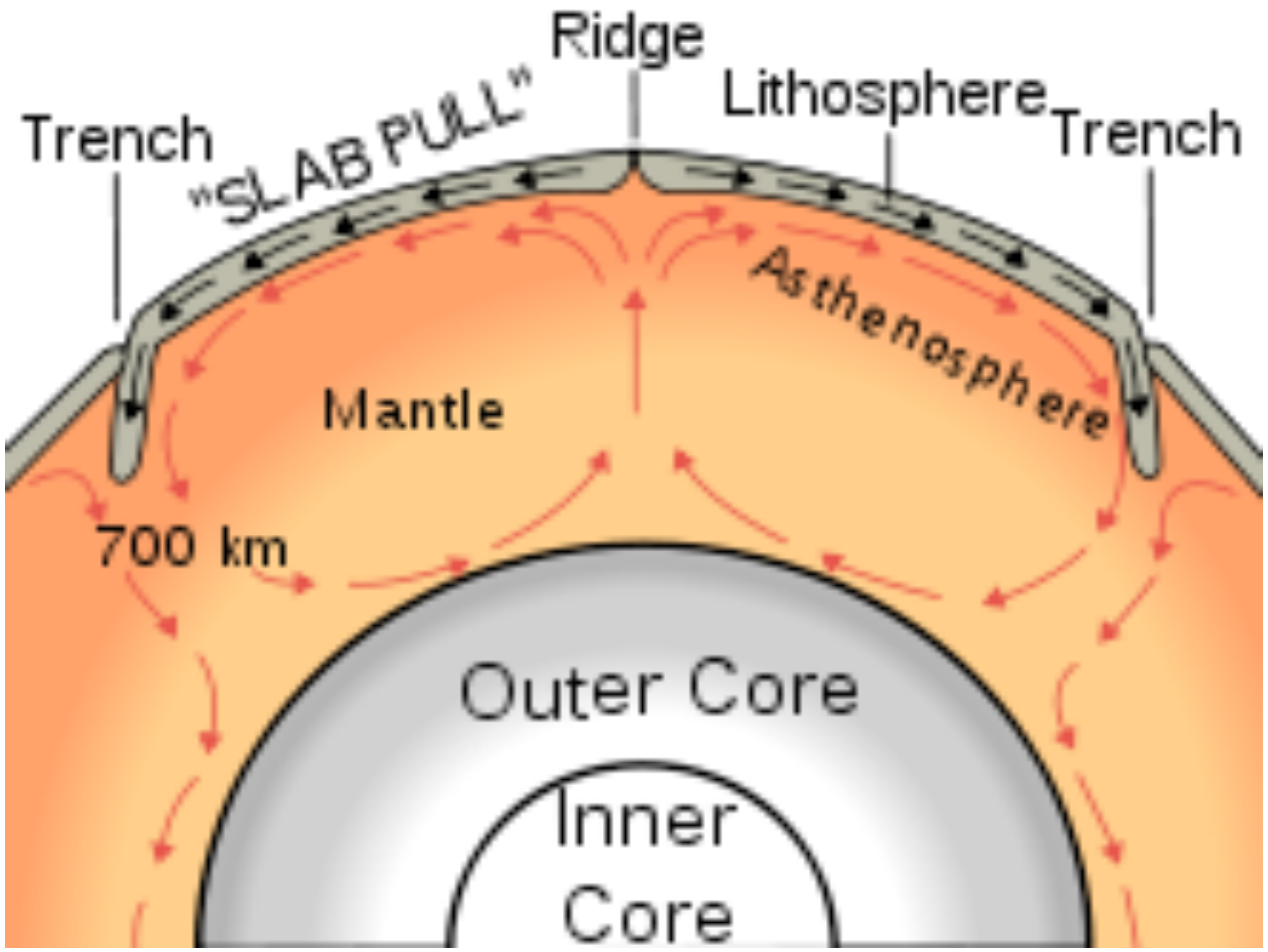
A convecting Earth

Plate tectonics provides a framework to understand global seismicity, volcanism and mountain building processes.

Plate motion and the interaction of these plates underpin this theory

**** Earth has a heat source coming from the core***

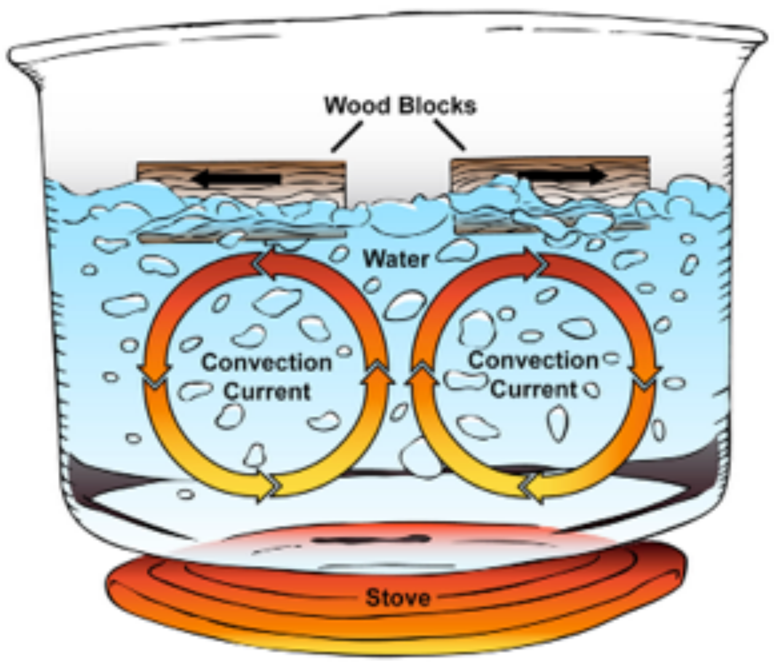
**** Heat is converted into motion via density variations***

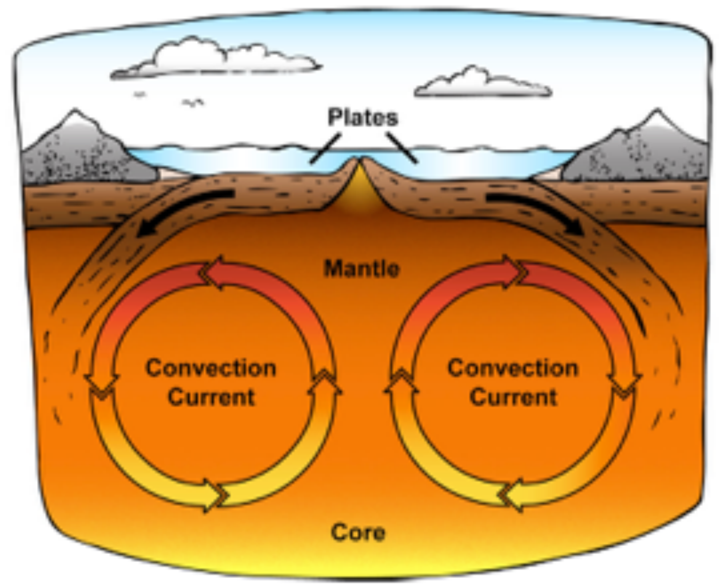


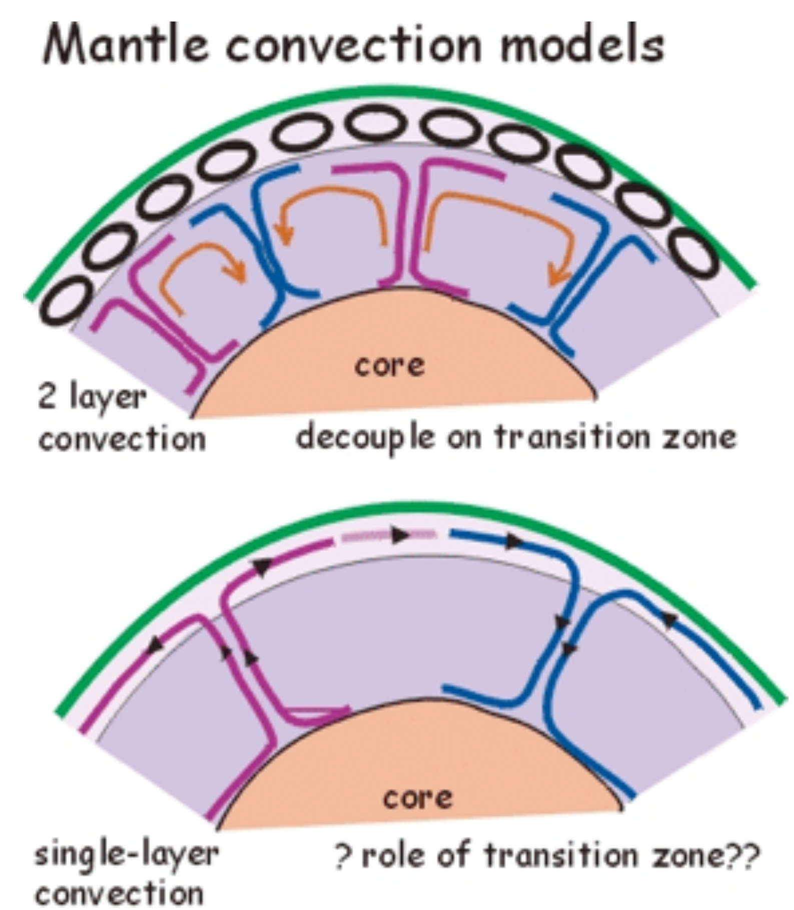
* Drifting continents
* Subduction zones
* Hotspots
* Spreading centres
* Post glacial rebound
* Heat budget

Thermal convection

http://dreamtigers.wordpress.com/2011/05/11/plate-tectonic-metaphor-illustrations-cmu/







http://asapscience.tumblr.com/post/50419005208/the-earths-center-is-out-of-sync-we-all-know

http://www.see.leeds.ac.uk/structure/dynamicearth/convection/models.htm

Variations in temperature cause small changes in the fluid density

Buoyancy forces cause cold (compressed, i.e. higher density) material to sink

Early experiments

Convection = flow driven by internal forces (buoyancy)

In the Earth, density variations are attributed to pressure, temperature and composition

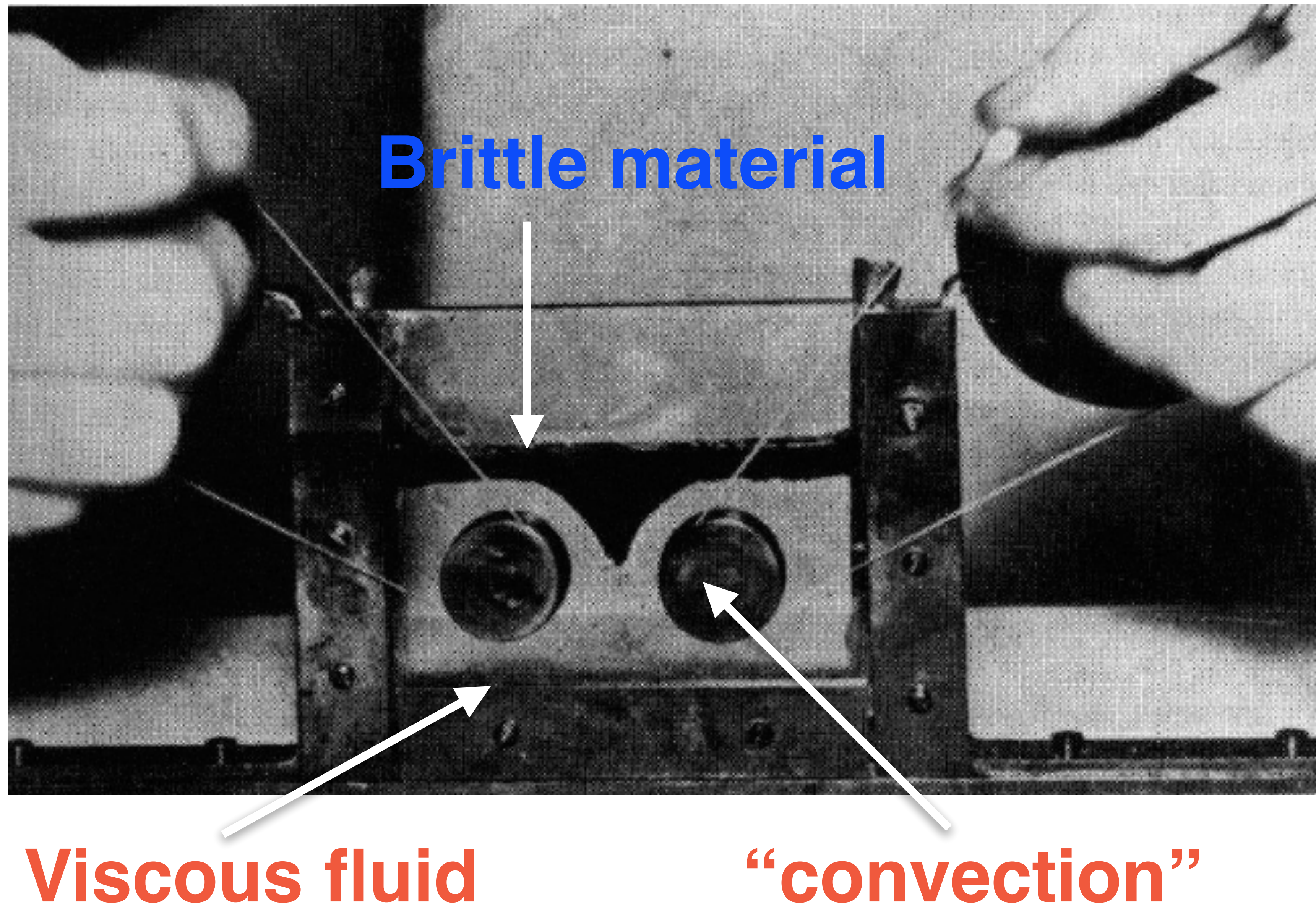
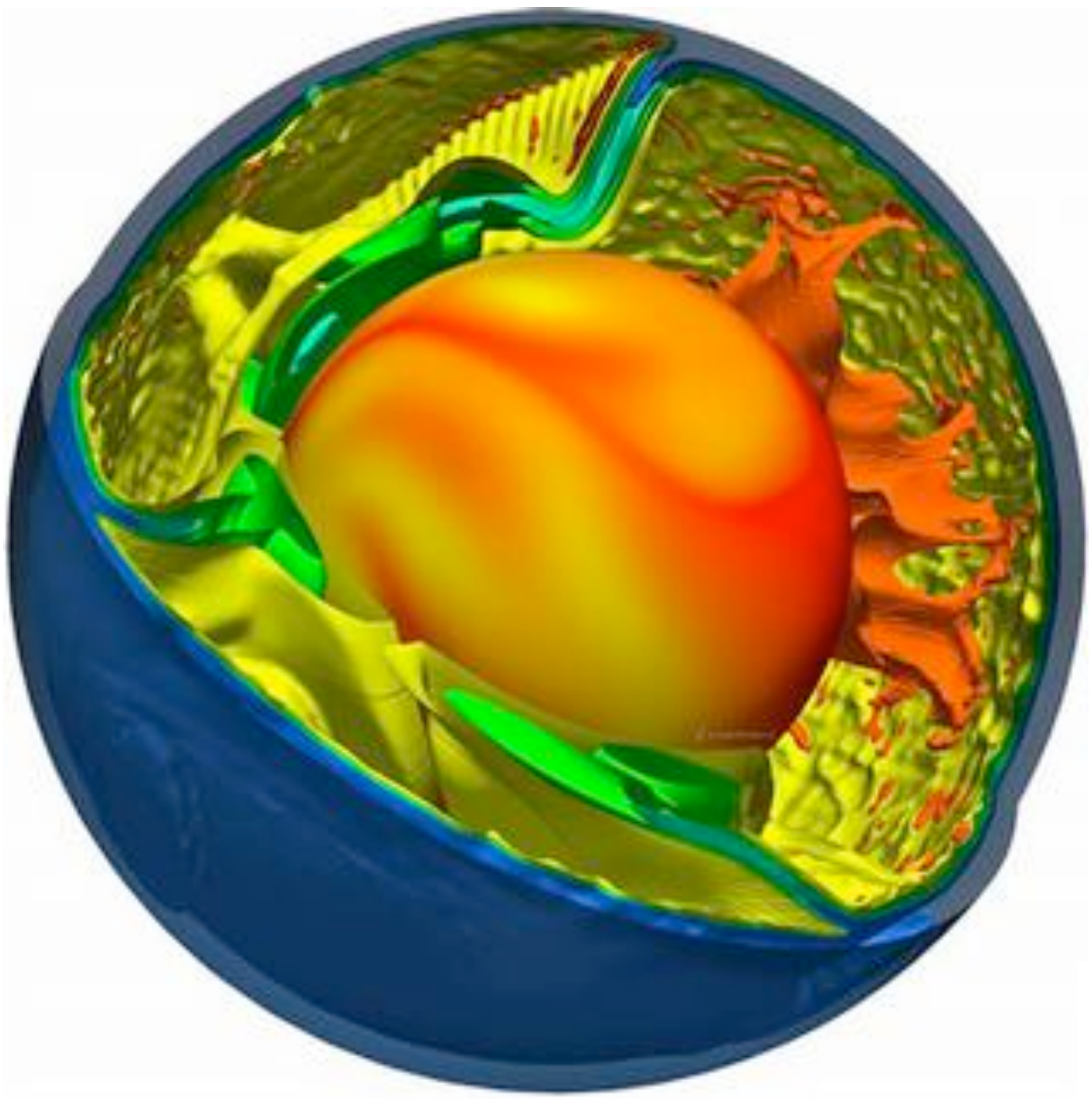
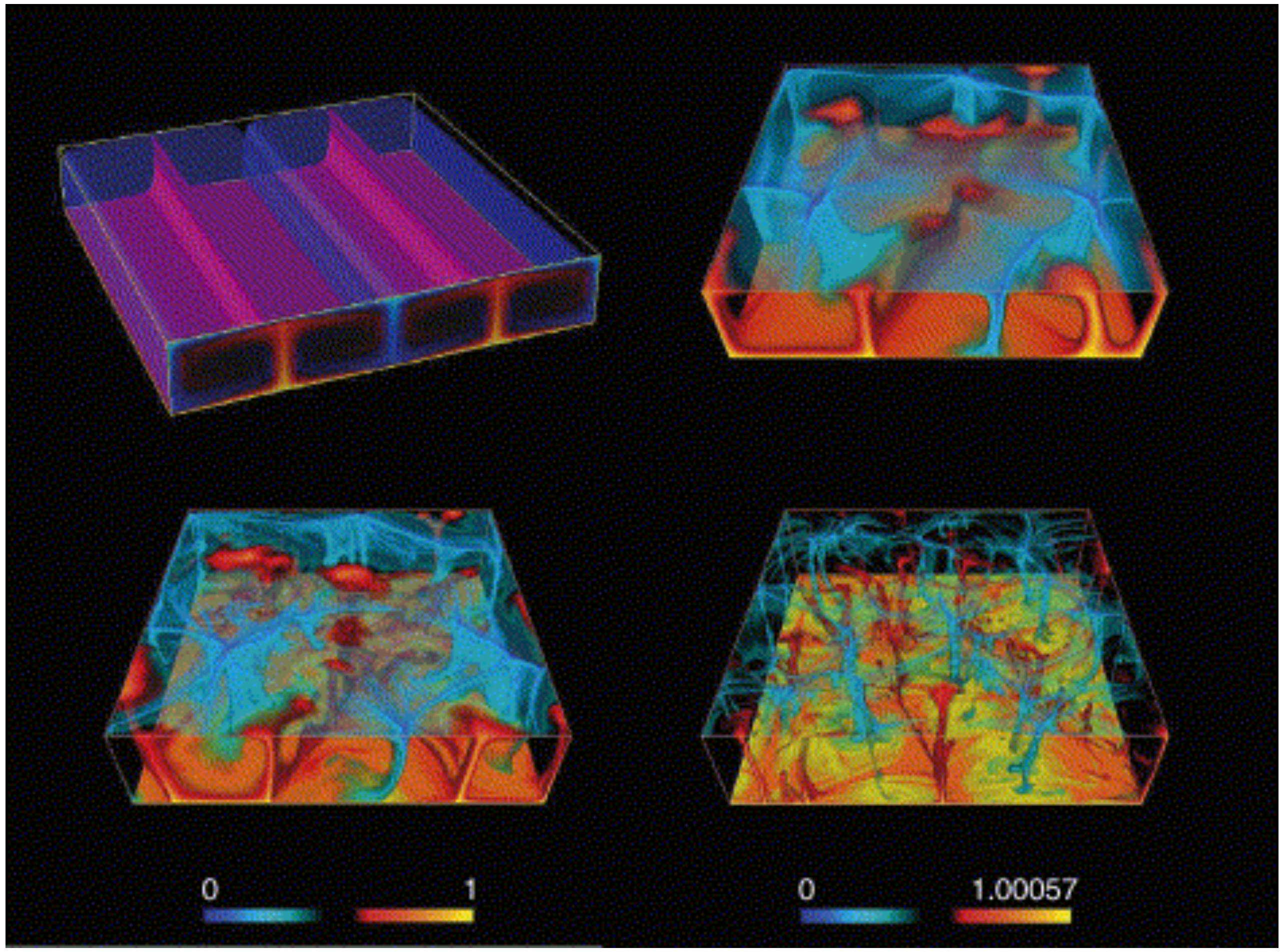


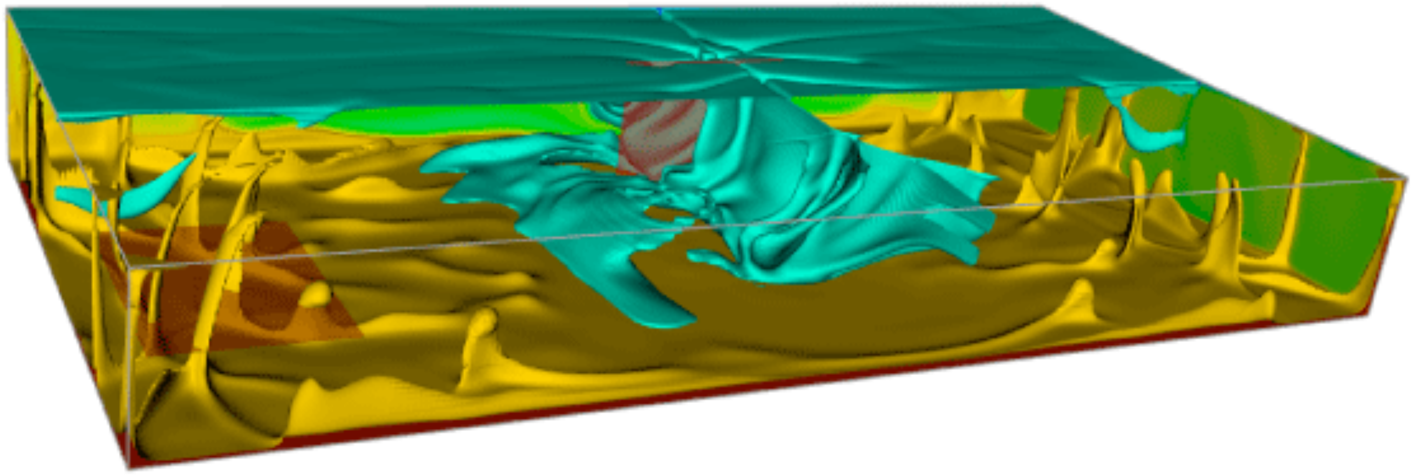
Figure 1.11. Early experiment on mantle convection by David Griggs (1939), showing styles of deformation of a brittle crustal layer overlying a viscous mantle. The cellular flow was driven mechanically by rotating cylinders.

Convection drives large-scale dynamics within in the Earth's mantle

Modern experiments







Fundamentals of fluid dynamics

Frames of reference

$$\frac{DT}{Dt} = \kappa \nabla^2 T$$

$$\frac{D}{Dt}$$ Material derivative

Stationary frame (Eulerian)

$$\frac{DT}{Dt} := \frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x}$$

→ *convective* rate of change due to transport of material to a different position (*x)* with respect to a fixed coordinate system

↓ *local* rate of change due to temporal variations at a position *x*

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \boldsymbol{v} \cdot \nabla T$$

Moving frame (Lagrangian)

$$\frac{DT}{Dt} := \frac{\partial T}{\partial t}$$

coordinate system is transported with the material, thus only the local rate of change remains

Conservation of mass

- General continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

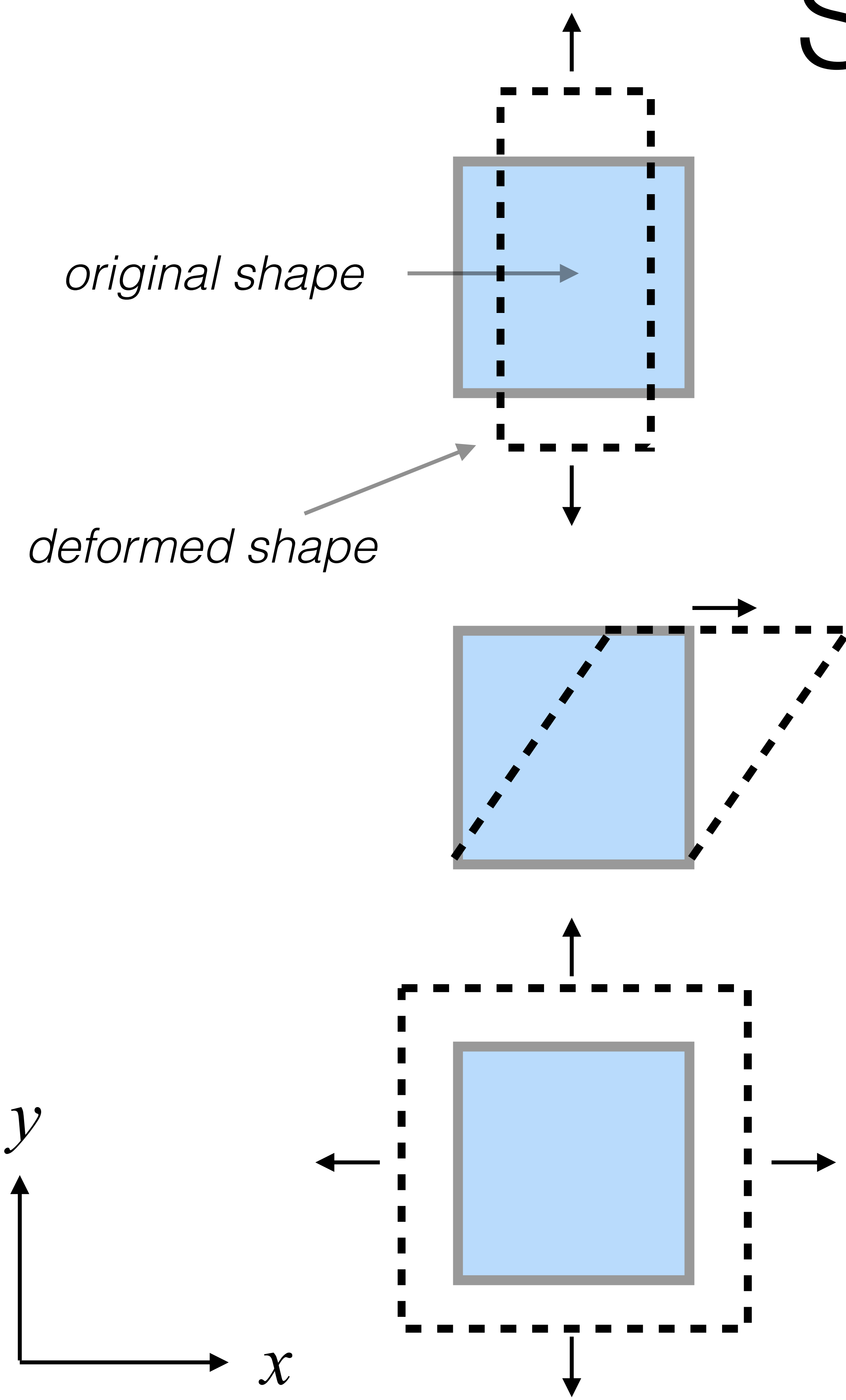
- Incompressible fluid

For many geological material (such as the Earth's crust and mantle), over long time-spans, one may assume an incompressible condition (e.g. the density of each material point does not change with time)

$$\frac{D\rho}{Dt} = 0 \quad \longrightarrow \quad \nabla \cdot \boldsymbol{v} = 0$$

Valid assumption when pressure and temperature changes are not very large and no phase transformations (e.g. no large volume changes) occur within the medium

Strain



Tensile strain - or - direct strain

$$\epsilon_{xx}, \epsilon_{yy}$$

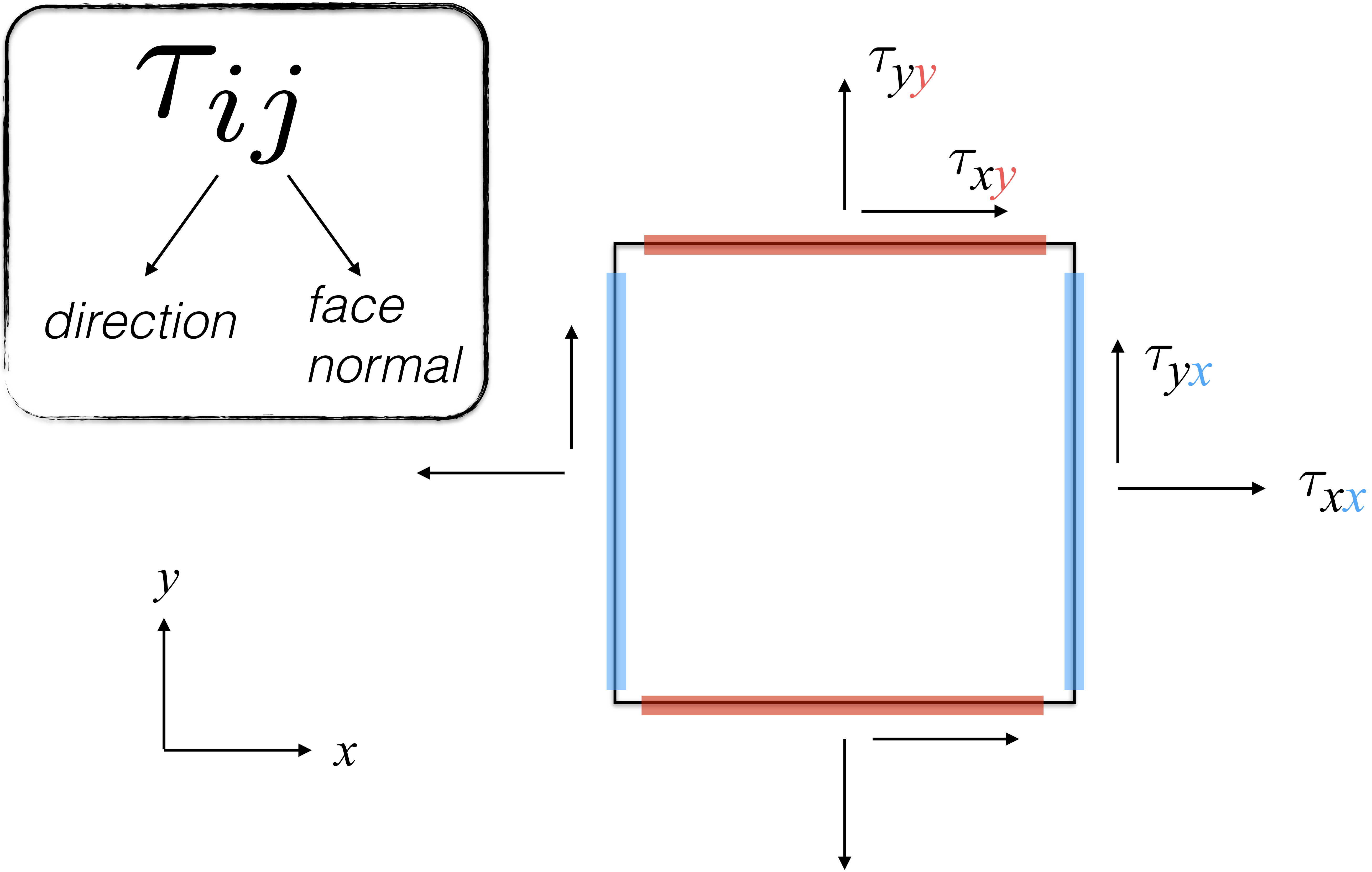
Shear strain

$$\epsilon_{xy}$$

Volumetric strain (dilatation)

$$\epsilon_{xx} + \epsilon_{yy}$$

Tensor: Physical meaning



Strain-rate tensor

- Time rate of change of the strain *tensor*

$$\dot{\epsilon}_{xx}, \dot{\epsilon}_{yy}, \dot{\epsilon}_{xy} \quad \text{[1/s]}$$

$\boldsymbol{x} = (x, y)$ position

$\boldsymbol{v} = (v_x, v_y)$ velocity

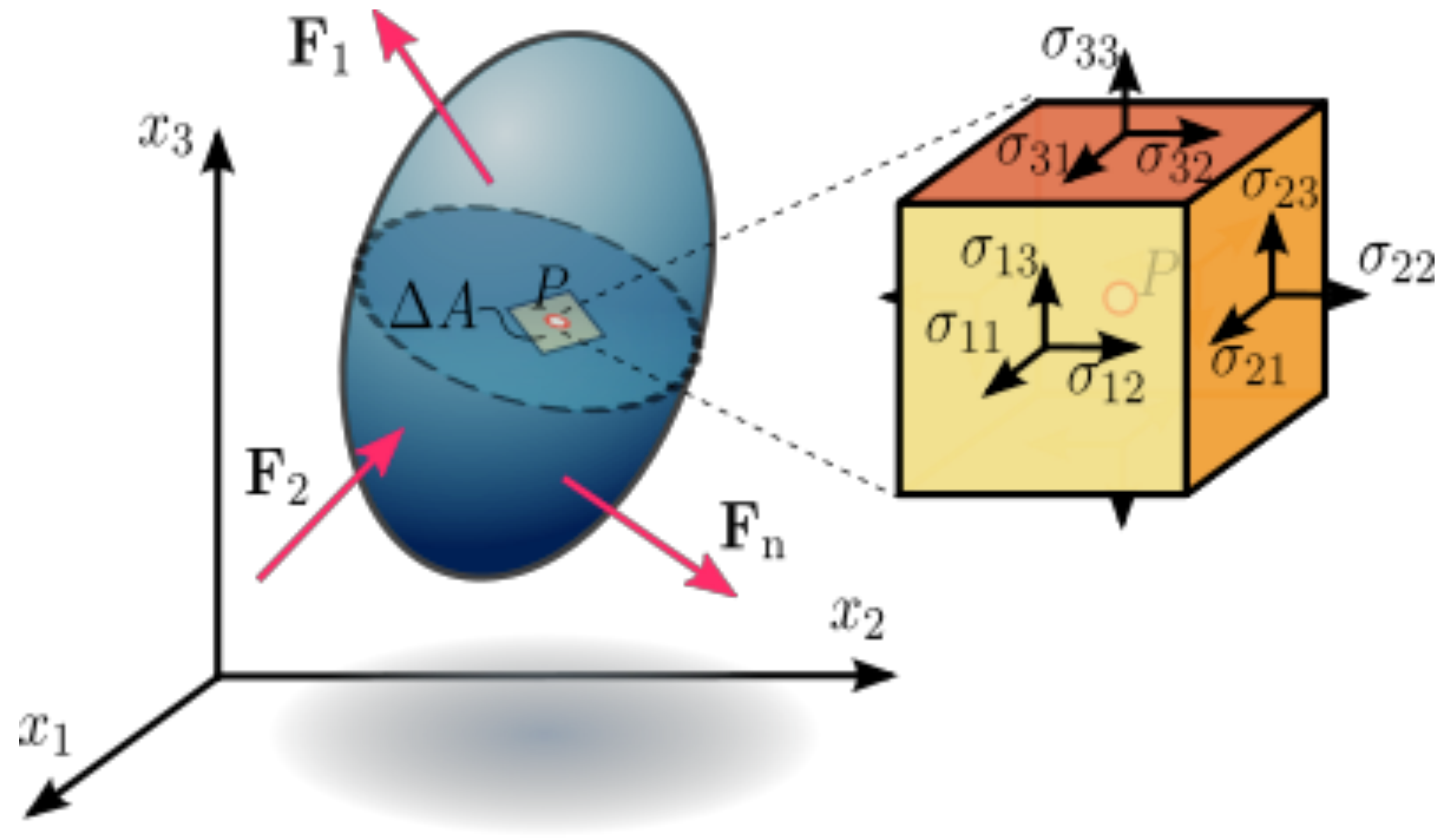
$$\dot{\epsilon}_{xx} = \frac{\partial v_x}{\partial x} \qquad \dot{\epsilon}_{yy} = \frac{\partial v_y}{\partial y} \qquad \dot{\epsilon}_{xy} = \frac{1}{2}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)$$

$$\dot{\epsilon}_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

- A *deviatoric* tensor has a mean (average) of zero, e.g.

$$\frac{1}{2}(\dot{\epsilon}_{xx} + \dot{\epsilon}_{yy}) = 0 \quad \Leftrightarrow \quad \frac{1}{2}\left(\sum_{k=1}^{2} \dot{\epsilon}_{kk}\right) = 0 \quad \Leftrightarrow \quad \frac{1}{2}\dot{\epsilon}_{kk} = 0$$

Stress tensor



σ_{ij} stress [Pa]

p pressure [Pa]

$$p = -\sigma_{kk} = -\frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}), \text{ for } k = 1, 2, 3$$

τ_{ij} deviatoric stress [Pa]

$$\sigma_{ij} = \tau_{ij} - p\delta_{ij}$$

stress = deviatoric stress - pressure

Constitutive law for a fluid

- Viscous stress

$$\tau_{ij} = 2\eta\,\tilde{\dot{\epsilon}}_{ij}$$

τ_{ij} → ***deviatoric stress* [Pa]**

η → ***viscosity* [Pa s]**

$\tilde{\dot{\epsilon}}_{ij}$ → ***deviatoric strain rate* [1/s]**

- Incompressible

$$\tau_{ij} = 2\eta\,\dot{\epsilon}_{ij} \longrightarrow$$ ***strain rate***

- Expanded form (2D)

$$\tau_{xx} = 2\eta\,\dot{\epsilon}_{xx} = 2\eta\,\frac{\partial v_x}{\partial x}$$

$$\tau_{zz} = 2\eta\,\dot{\epsilon}_{zz} = 2\eta\,\frac{\partial v_z}{\partial z}$$

$$\tau_{xz} = 2\eta\,\dot{\epsilon}_{xz} = \eta\,\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right)$$

Conservation of momentum

- General equation: $$\rho\left(\frac{\partial v_i}{\partial t} + v_k\,\frac{\partial v_i}{\partial x_k}\right) = \frac{\partial \sigma_{ij}}{\partial x_j} - \rho g_i$$

  Dimensional considerations:

  - plate velocity ~ 1 cm/year, changes occur on 1 Myr timescale → $\rho \partial v / \partial t \sim 10^{-19}$

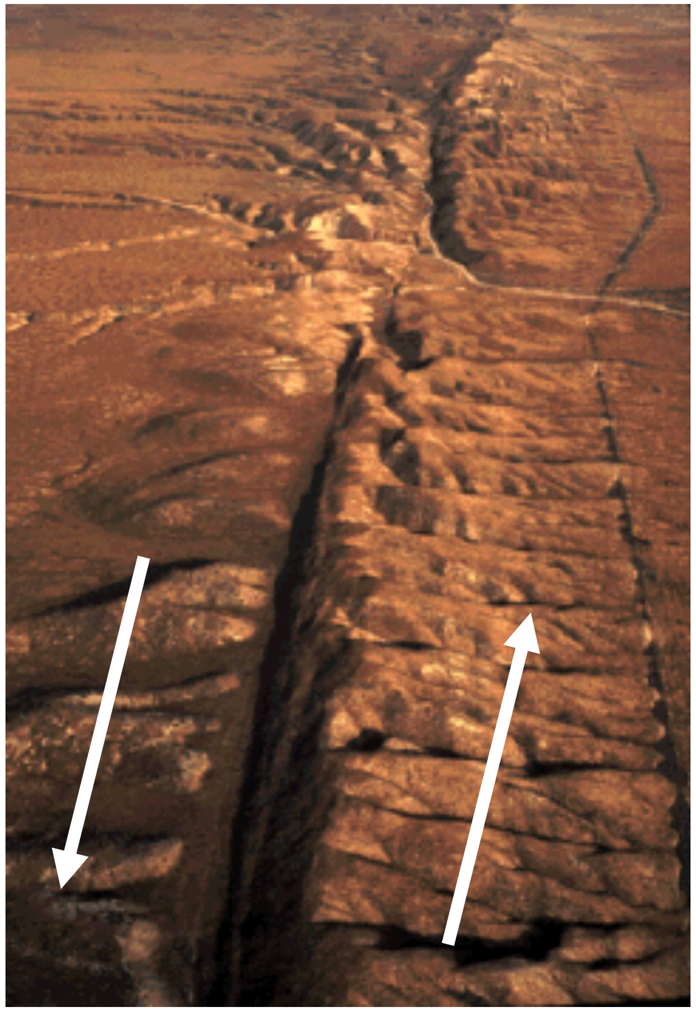
  - density ~ 3000 kg/m^3, g = 10 m/s^2 → $\rho g \sim 10^4$

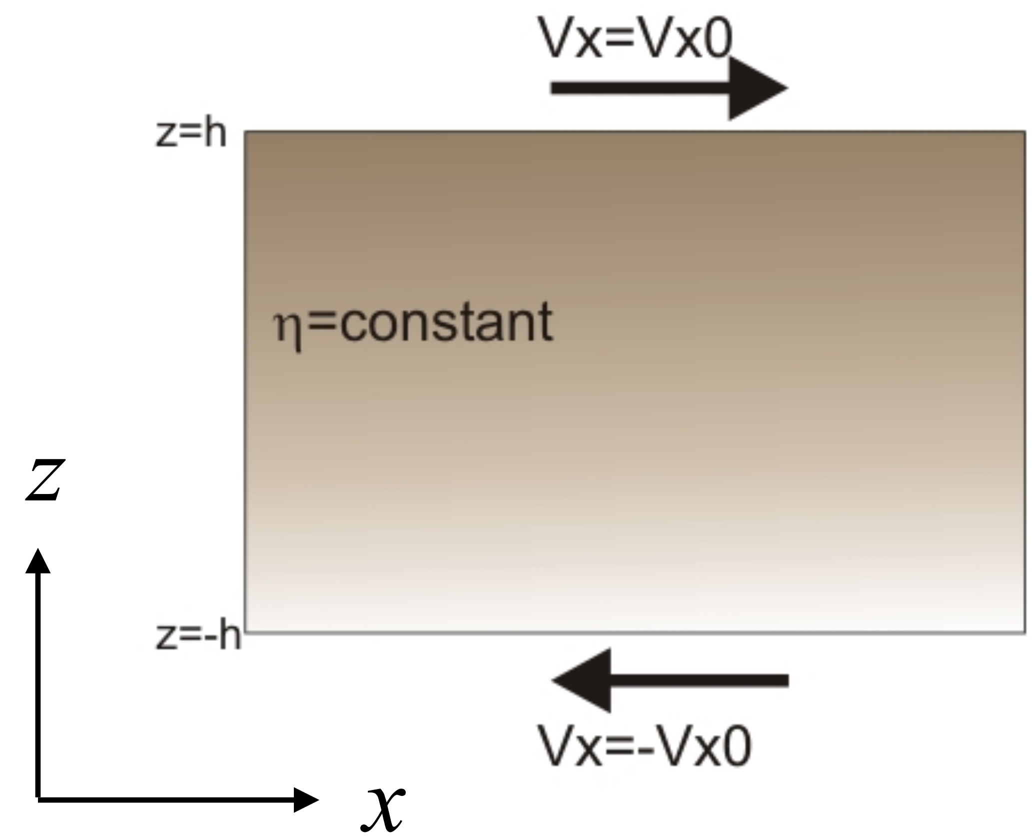
- Mom. balance for a creeping fluid: $$0 = \frac{\partial \sigma_{ij}}{\partial x_j} - \rho g_i$$

  “Stokes flow”

- Primitive equations (***v***,*p*) obtained by
  - inserting relationship for deviatoric stress and pressure
  - inserting constitutive law
  - replacing strain-rates by velocity gradients

1D shear flow (Couette)





- Ignore body forces
- Steady state
- Constant viscosity
- 1D velocity field
- Pressure gradient is zero

$$\longrightarrow \quad \sigma_{xz} = \eta \frac{V_x^0}{h}$$

With:

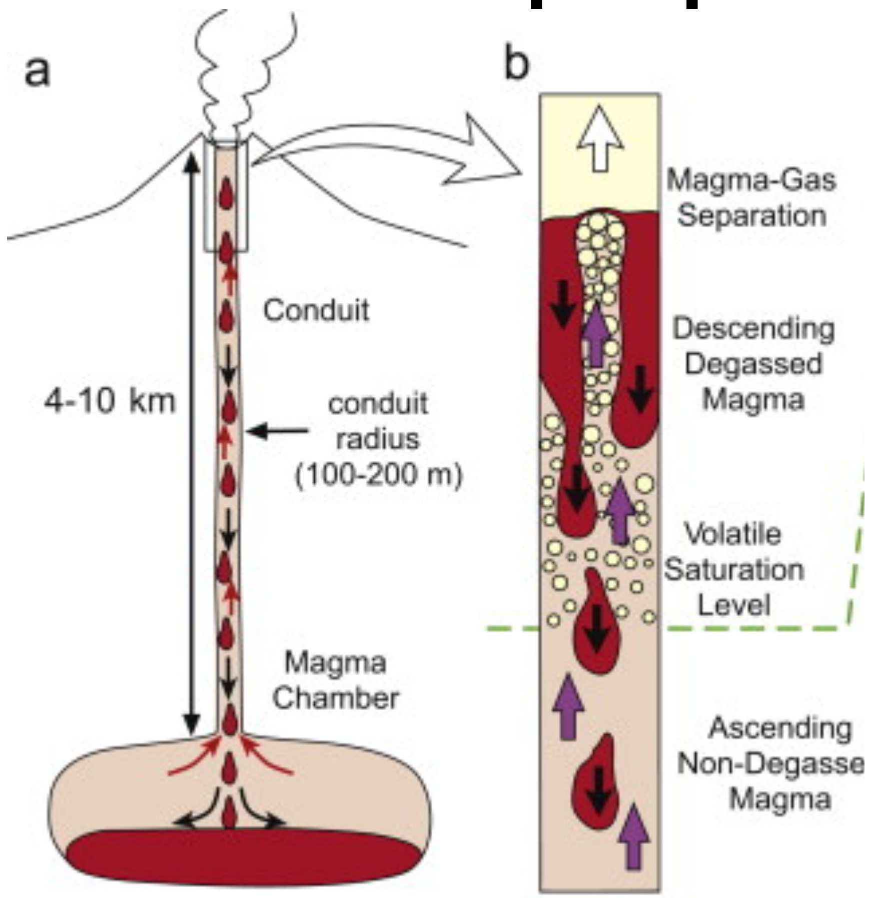
Vx = 5 cm/year

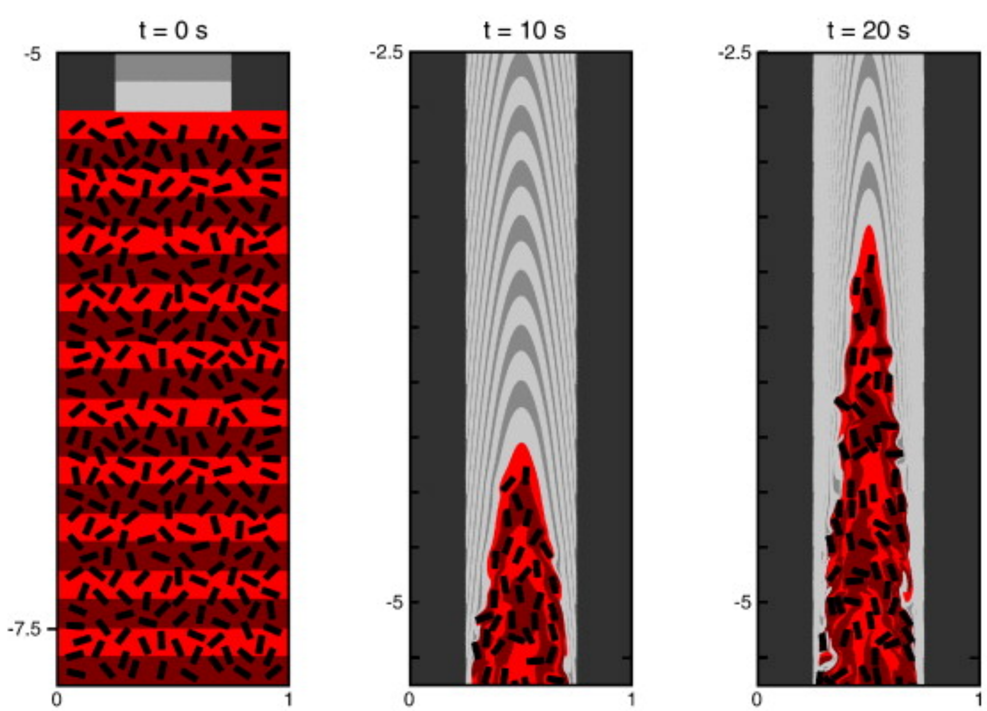
H ~100 km

η ~ 10^{22} Pa s

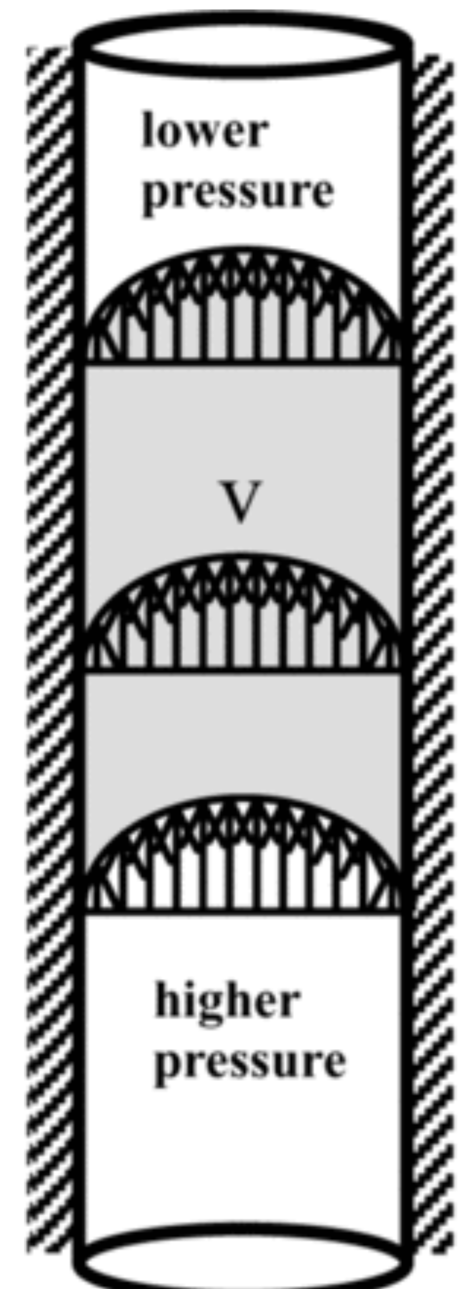
σ_{xz} ~ 150 MPa

1D pipe flow (Poiseuille)





- Ignore body forces
- Steady state
- 1D velocity field: $V_z(x)$
- Pressure gradient is constant
- Viscosity is constant



Dynamic similarity

- Perform an infinite number of experiments to understand the dynamics
- or… find a more compact representation of the equations
- Non-dimensional numbers

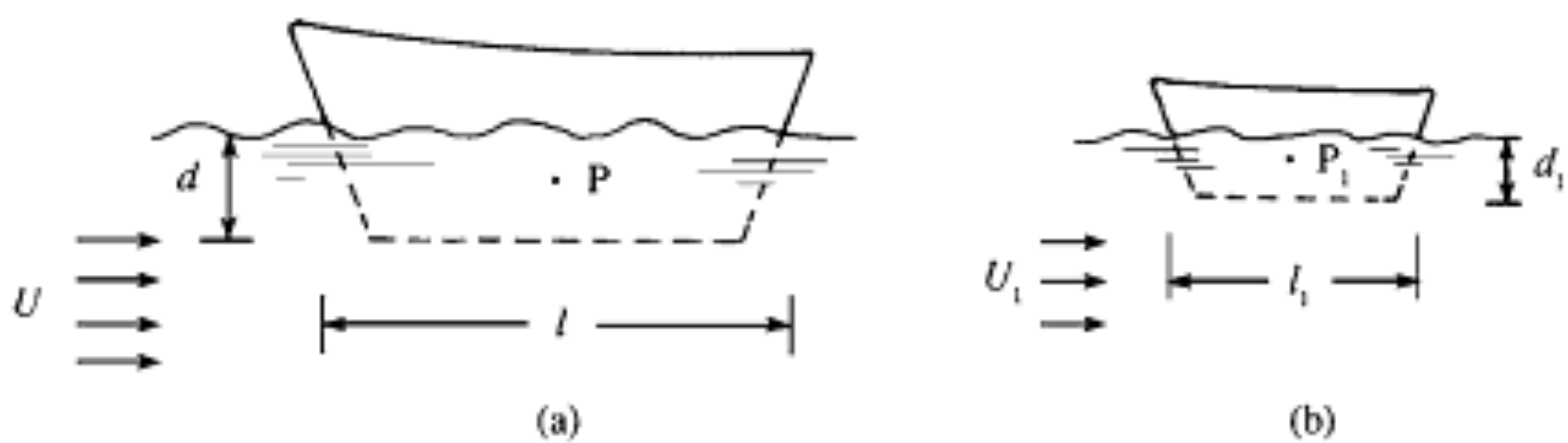
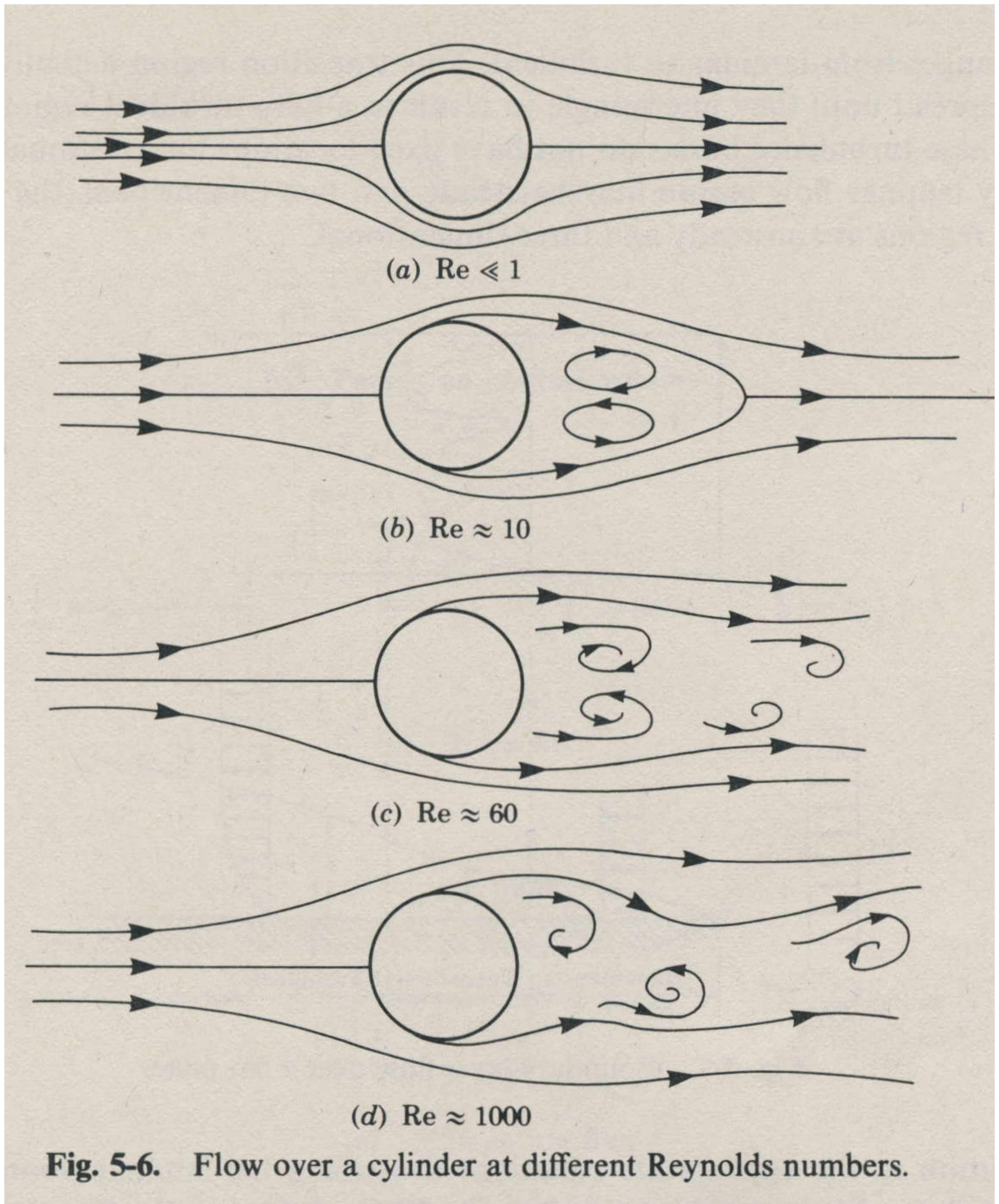
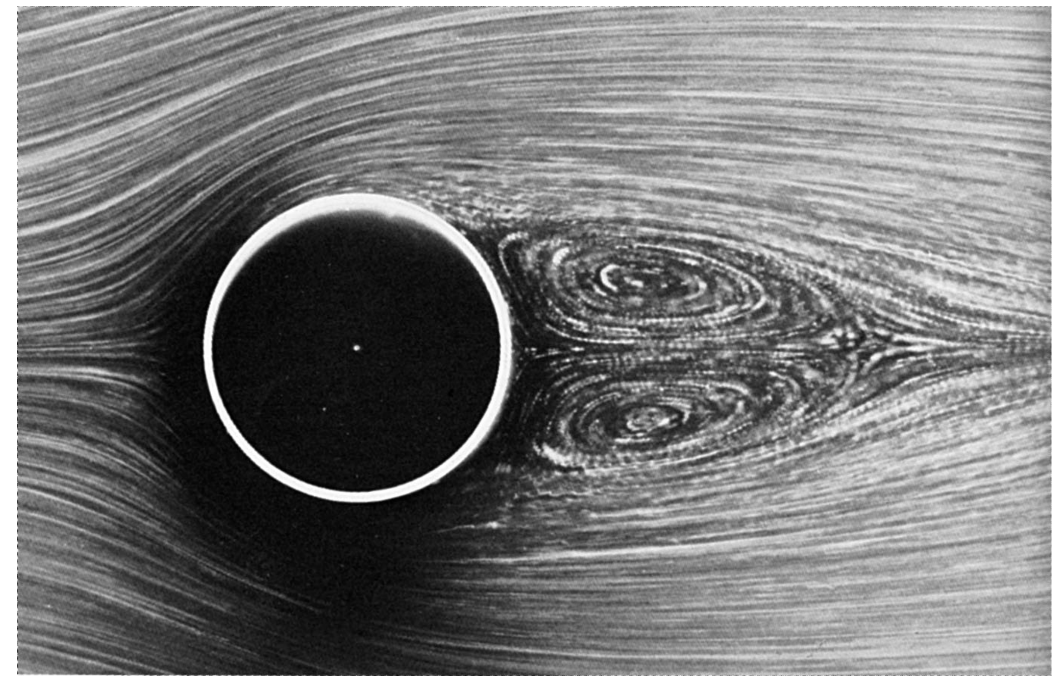
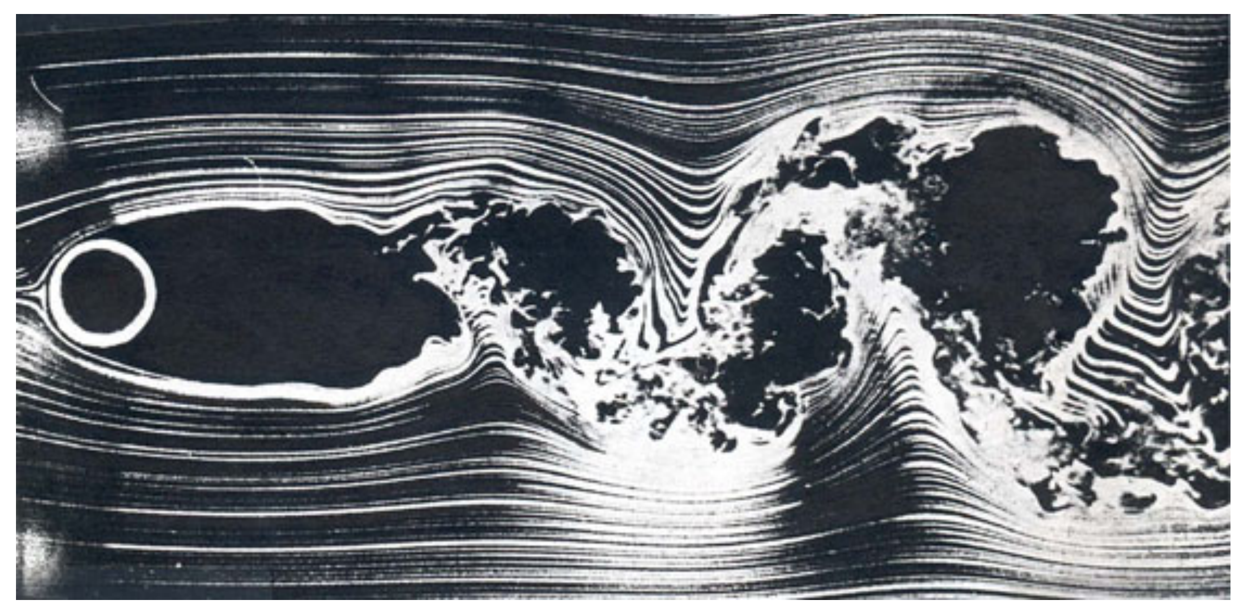


Figure 8.1 Two geometrically similar ships.

Flow regimes



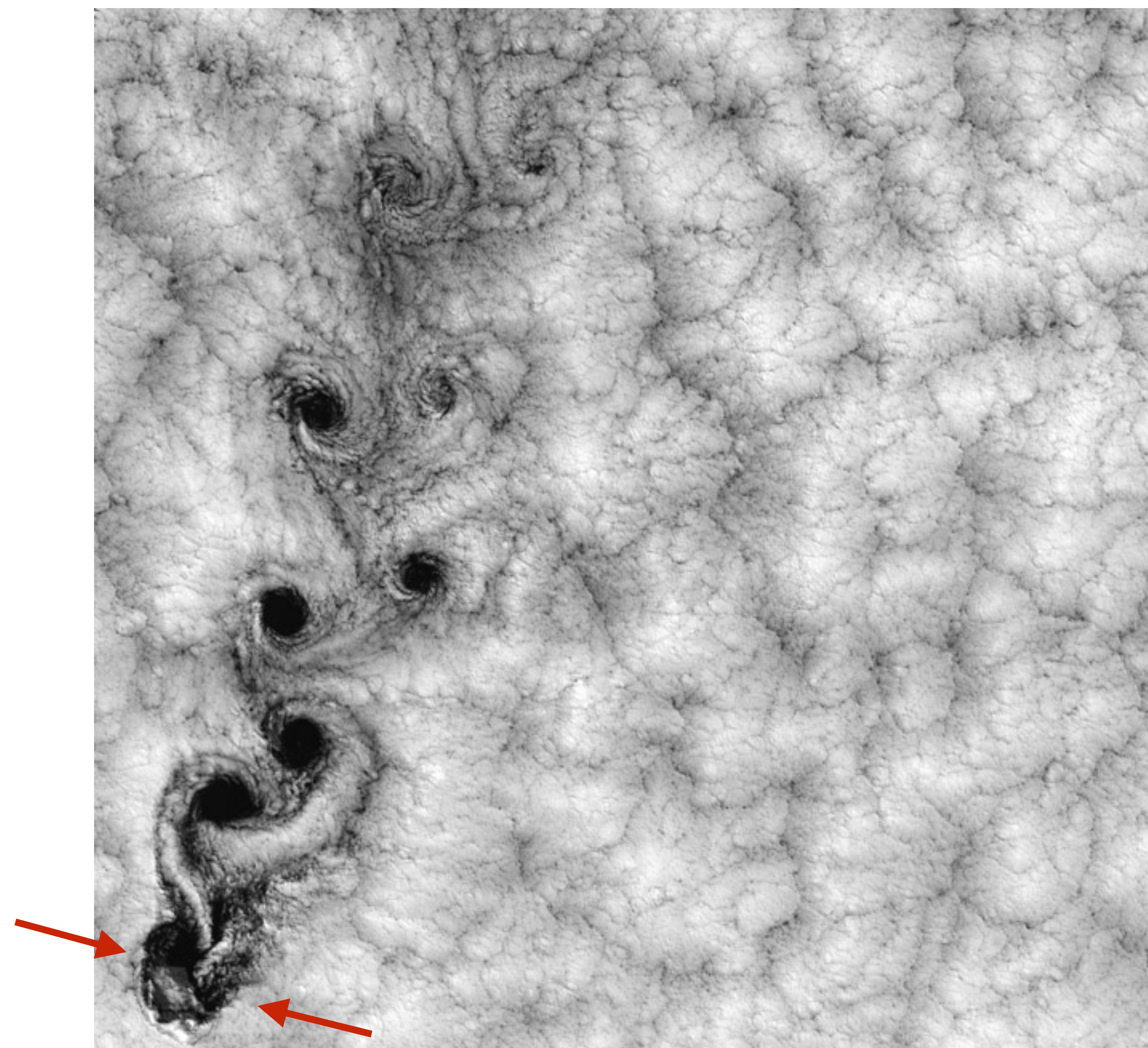




Flow regimes - Issue of scale

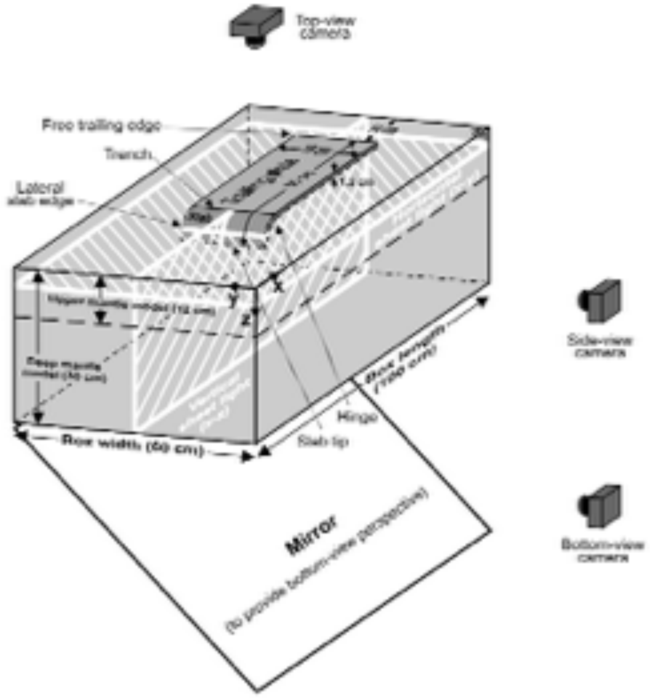
O(cm)

Karman vortex sheet caused by wind flowing around the Juan Fernandez Islands of the Chilean coast

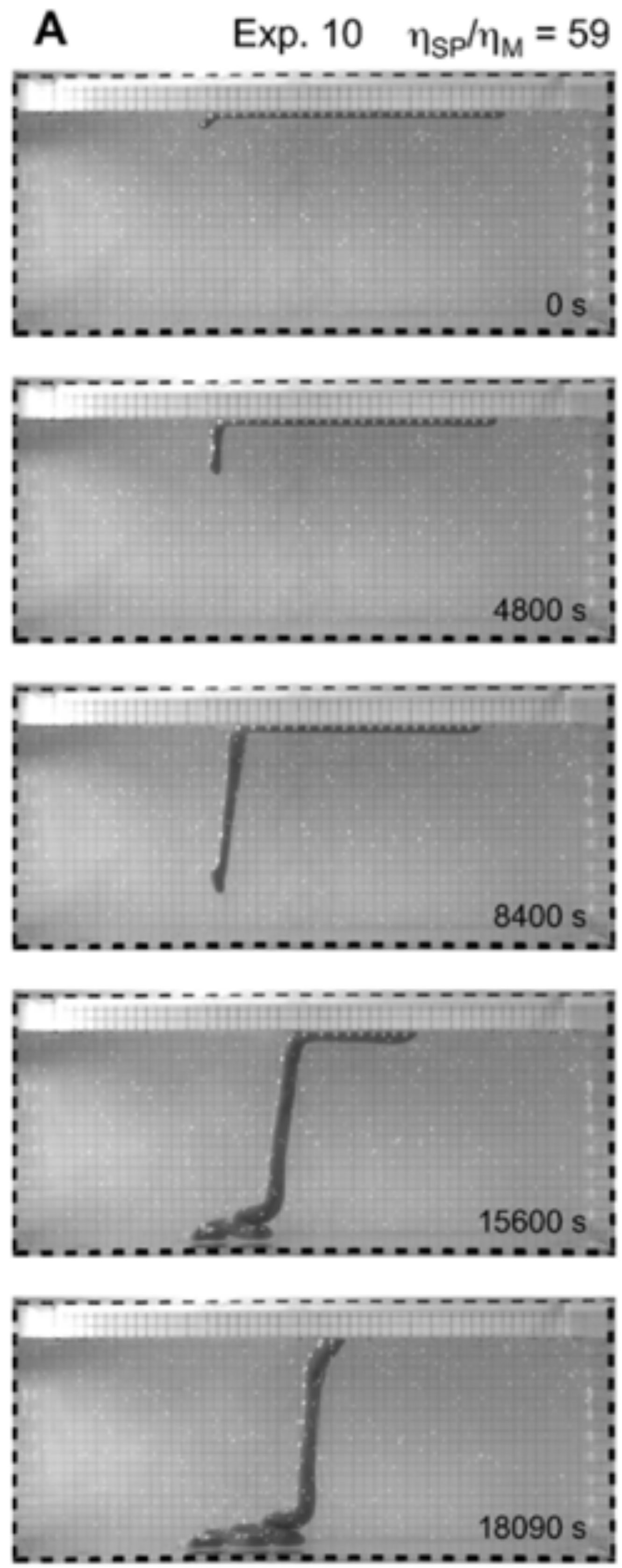


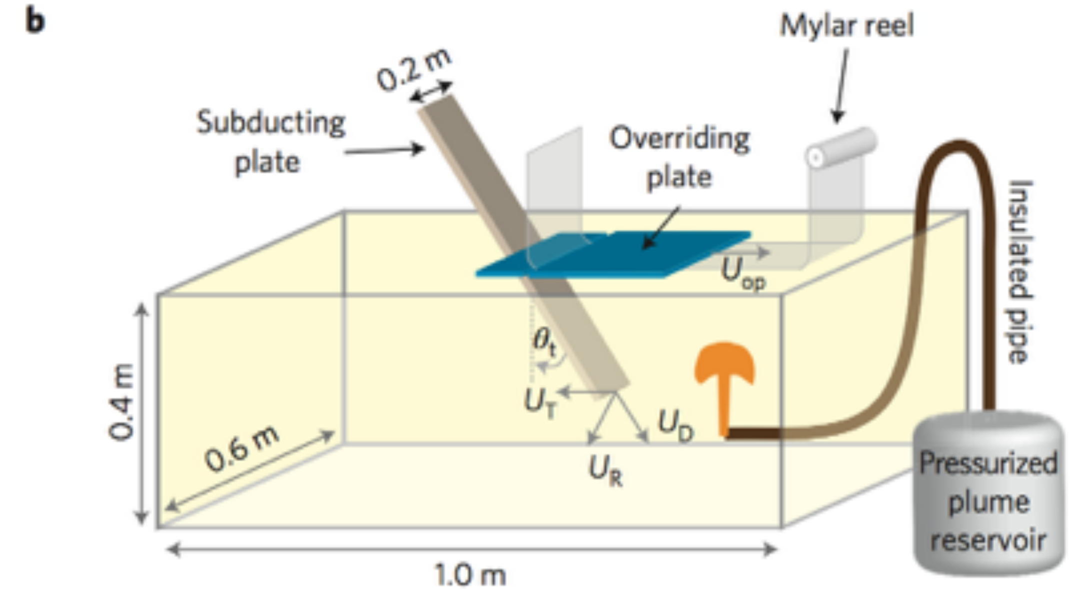
O(km)

Permits meaningful models to be developed

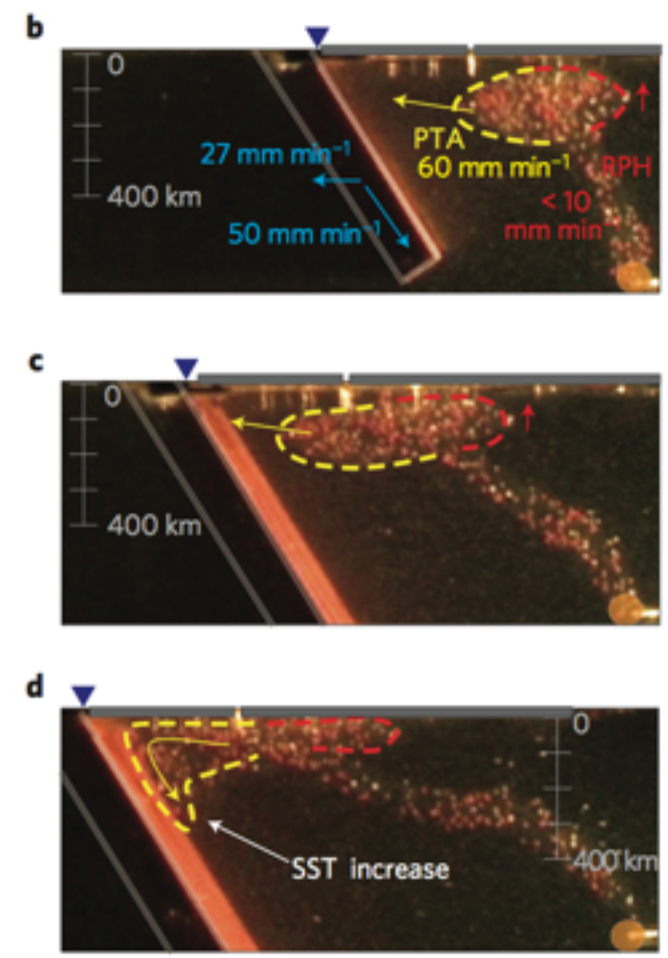


Schellart (2008)





Kincaid (2013)



Reynolds number of the mantle

$$x_i' = \frac{x_i}{h} \qquad u_i' = \frac{u_i}{U} \qquad t' = \frac{U}{h}t \qquad p' = \frac{1}{\rho U^2}p$$

$$Re = \frac{\rho h U}{\eta}$$ $\longrightarrow$ ratio of inertial forces to viscous forces

$$\frac{|\rho D u_u / Dt|}{|\eta \partial^2 u_i / \partial x_k \partial x_k|} = Re \frac{|D u_i' / D t'|}{|\partial^2 u_i' / \partial x_k' \partial x_k'|}$$

Characteristic values for the mantle

ρ ~ ? kg/m3

U ~ ? m/s

h ~ ? km

η ~ $10^?$ Pas

Re ~ ?

Fundamentals of mantle convection

(buoyancy driven incompressible Stokes flow)

Boussinesq approximation

- Thermal variations in a fluid lead to small amounts of expansion / contraction.
- Expansion results in lowering of density, e.g. resulting in a buoyancy force —> leading to fluid motion

$$\rho = \rho_0 + \rho' \xrightarrow{\text{perturbation}} \rho' \ll \rho_0$$

ρ_0 → Reference density

$$\rho' = -\rho_0 \alpha_v \left(T - T_0\right)$$

T_0 → Reference temperature corresponding to ref. density

α_v → coefficient of thermal expansivity [1/K]

Mantle convection eqns.

Compact form:

$$-\nabla p + \nabla \cdot \eta \left(\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^T \right) = \rho_0 \left(1 - \alpha (T - T_0) \right) g \hat{\boldsymbol{e}}_z$$

$$\nabla \cdot \boldsymbol{v} = 0$$

$$\rho C_p \frac{DT}{Dt} = \nabla \cdot (k \nabla T) + \rho Q$$

Constant viscosity and constant conductivity:

$$-\nabla p + \eta \nabla^2 \boldsymbol{v} = \rho_0 \left(1 - \alpha (T - T_0) \right) g \hat{\boldsymbol{e}}_z$$

$$\nabla \cdot \boldsymbol{v} = 0$$

$$\frac{DT}{Dt} = \kappa \nabla^2 T + \frac{Q}{C_p}$$

Non-dimensional form

(1) Dimensional form:

$$-\nabla p + \eta \nabla^2 \boldsymbol{v} = \rho_0 \left(1 - \alpha(T - T_0)\right) g \hat{\boldsymbol{e}}_z$$
$$\nabla \cdot \boldsymbol{v} = 0$$
$$\frac{DT}{Dt} = \kappa \nabla^2 T$$

(2) Perturbation from background state:

$$-\nabla p + \eta \nabla^2 \boldsymbol{v} = -\rho_0 \alpha T g \hat{\boldsymbol{e}}_z$$
$$\nabla \cdot \boldsymbol{v} = 0$$
$$\frac{DT}{Dt} = \kappa \nabla^2 T$$

(3) Scaling:

$$\boldsymbol{x}' = \boldsymbol{x}/h$$
$$t' = t/(h^2/\kappa)$$
$$T' = T/\Delta T$$
$$\boldsymbol{v}' = \boldsymbol{v}/(\kappa/h)$$
$$p' = p/(\eta\kappa/h^2)$$

(4) Non-dimensional form:

$$-\nabla' p' + \nabla'^2 \boldsymbol{v}' = -Ra T' \hat{\boldsymbol{e}}_z$$
$$\nabla' \cdot \boldsymbol{v}' = 0$$
$$\frac{DT'}{Dt'} = \nabla'^2 T'$$

$$\longrightarrow \quad Ra = \frac{\alpha \rho_0 g \Delta T h^3}{\eta \kappa}$$

Rayleigh number

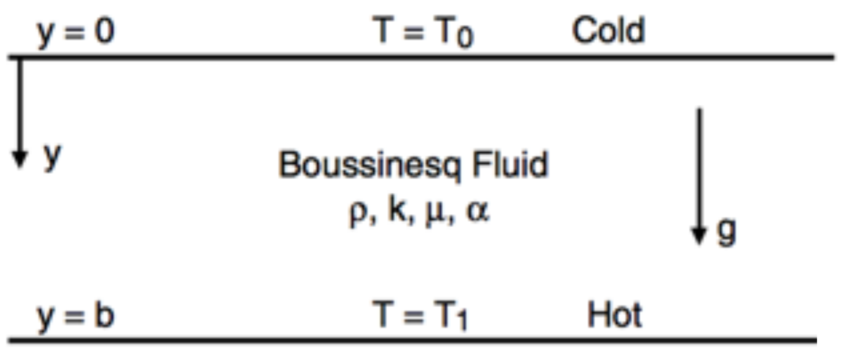
Rayleigh number (*Ra*)

- Describes the relationship between buoyancy and viscosity within a fluid
- The Rayleigh number gives a sense of the “vigour” of convection
- For small temperature difference (small *Ra*): conduction is the dominant mode of heat transfer
- For larger *Ra*: convection becomes the dominant mode
- The transition between conduction and convection is referred to as the “critical Rayleigh number (Ra_{cr})”
- Where does this transition occur?

Stability analysis

Model definition:

Figure 7.1. Sketch of a plane fluid layer heated from below for the problem of the onset of thermal convection.



Procedure:

$$T = T_{\text{conductive}} + T' = T_0 + T_1$$

$$T_1(x, y) = f(y) \sin\left(\frac{2\pi x}{\lambda}\right)$$

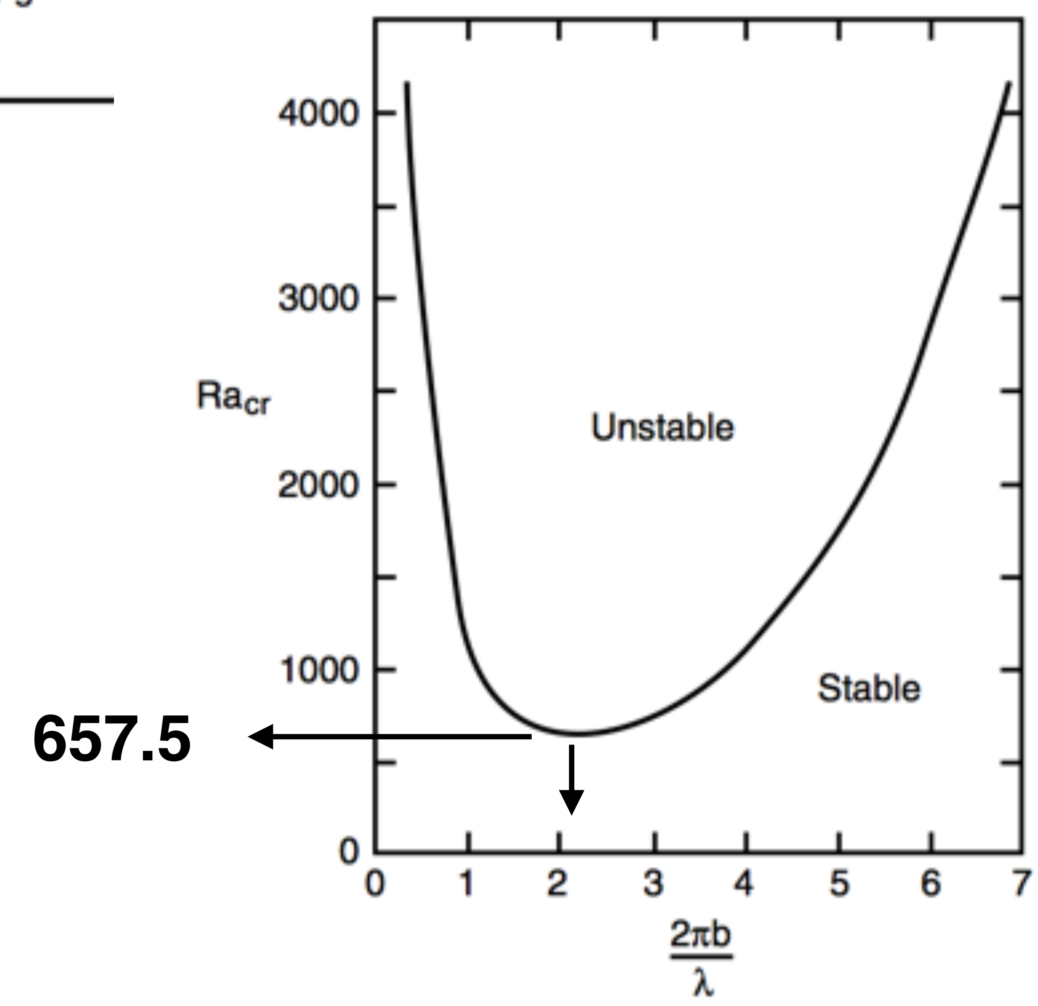
* For small velocity and T_1, we can linearise the system

* Energy equation expressed in terms of T_1 and V_z

* “Small” terms are neglected

* Solve via separation of variables

$$\longrightarrow \quad Ra_{cr} = \left(\pi^2 + \frac{4\pi^2 b^2}{\lambda^2}\right)^3 \left(\frac{4\pi^2 b^2}{\lambda^2}\right)^{-1}$$



Below *Ra* critical, no convection occurs

Critical Rayleigh number

- More than one definition exists…

Table 7.1. ***Values of the Minimum Critical Rayleigh Number and Associated Dimensionless Horizontal Wavelength for the Onset of Convection in Plane Fluid Layers with Different Surface Boundary Conditions and Modes of Heating***

Surface Boundary Conditions and Mode of Heating	Ra_{cr} (min)	λ^*_{cr}
Both boundaries shear stress free and isothermal, no internal heating. $H^* = 0$.	657.5	$2\sqrt{2} = 2.828$
Both boundaries fixed and isothermal. $H^* = 0$.	1,707.8	2.016
Shear stress free upper boundary, fixed lower boundary, both boundaries isothermal. $H^* = 0$.	1,100.7	2.344
Both boundaries shear stress free, upper boundary isothermal, lower boundary specified heat flux. $H^* = 0$.	384.7	3.57
Both boundaries fixed, upper boundary isothermal, lower boundary specified heat flux. $H^* = 0$.	1,295.8	2.46
Upper boundary shear stress free and isothermal, lower boundary fixed and heat flux prescribed. $H^* = 0$.	816.7	2.84

Numerical experiments

$$-\nabla' p' + \nabla'^2 \boldsymbol{v}' = -RaT'\hat{\boldsymbol{e}}_z$$

$$\nabla' \cdot \boldsymbol{v}' = 0$$

$$\frac{DT'}{Dt'} = \nabla'^2 T'$$

$$Ra = \frac{\alpha \rho_0 g \Delta T h^3}{\eta \kappa}$$

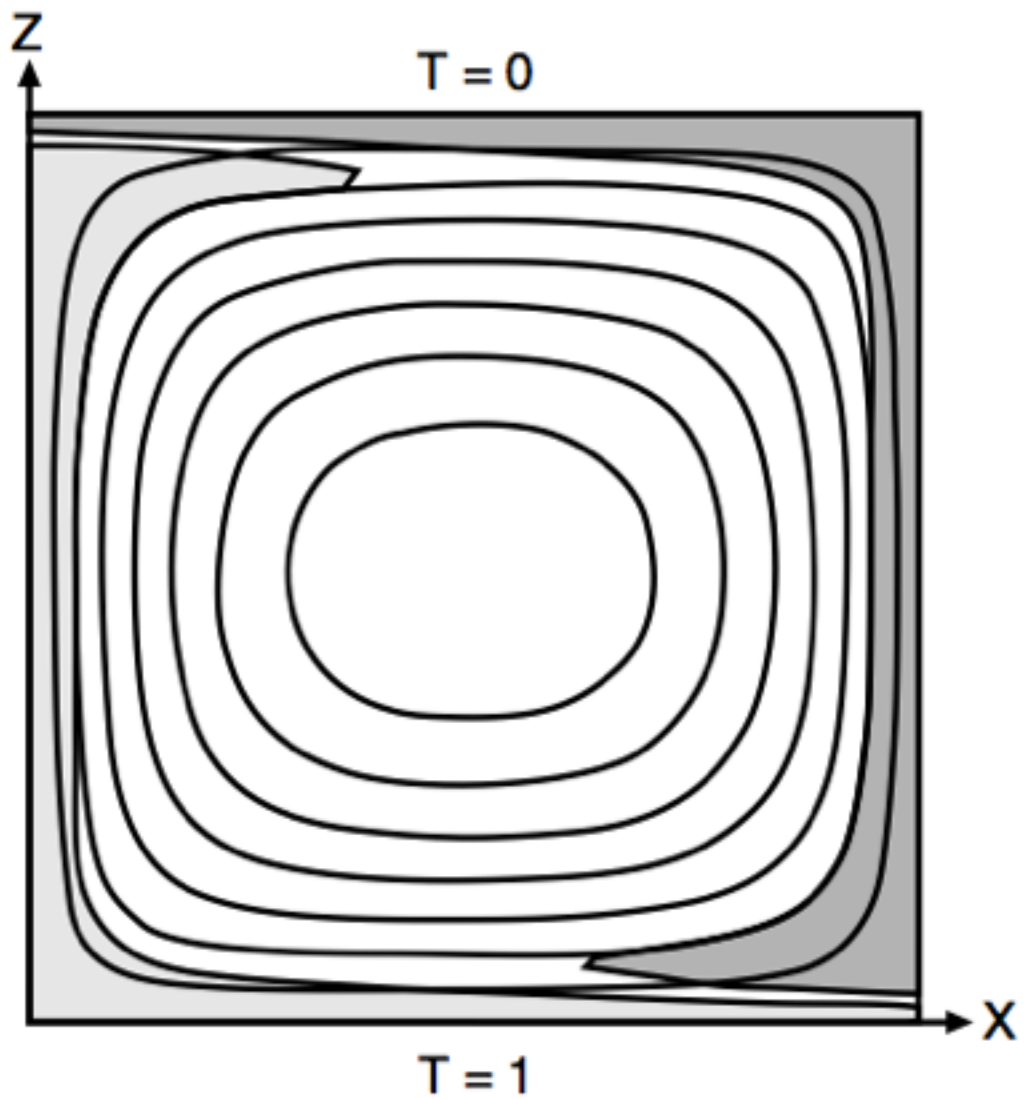


Figure 9.1. The structure of steady-state, two-dimensional, Rayleigh–Bénard convection at infinite Prandtl number, with streamlines of the motion (solid contours), hot thermal boundary layer and rising plume (light shading), and cold thermal boundary layer and sinking plume (dark shading).

Numerical experiments

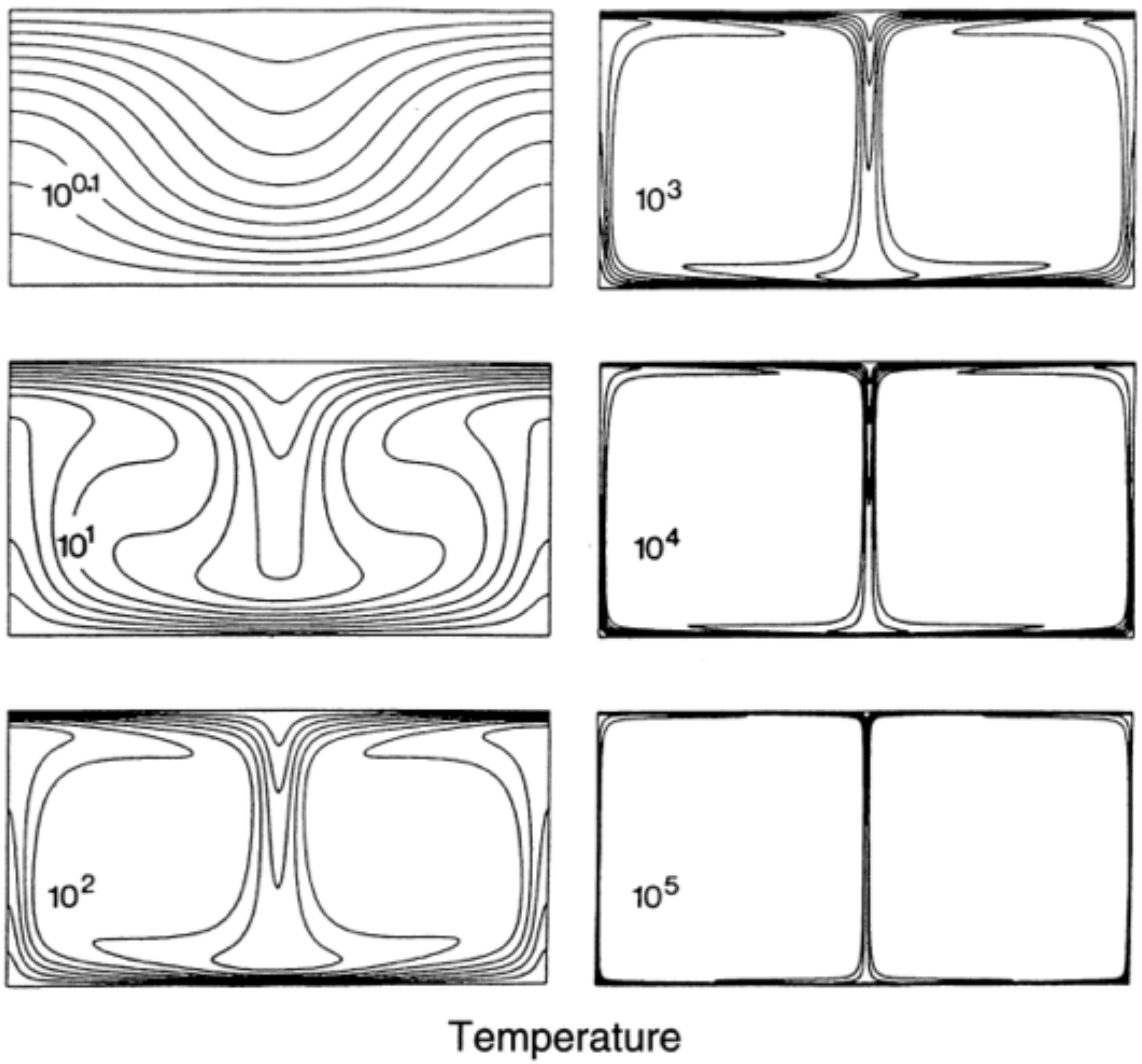
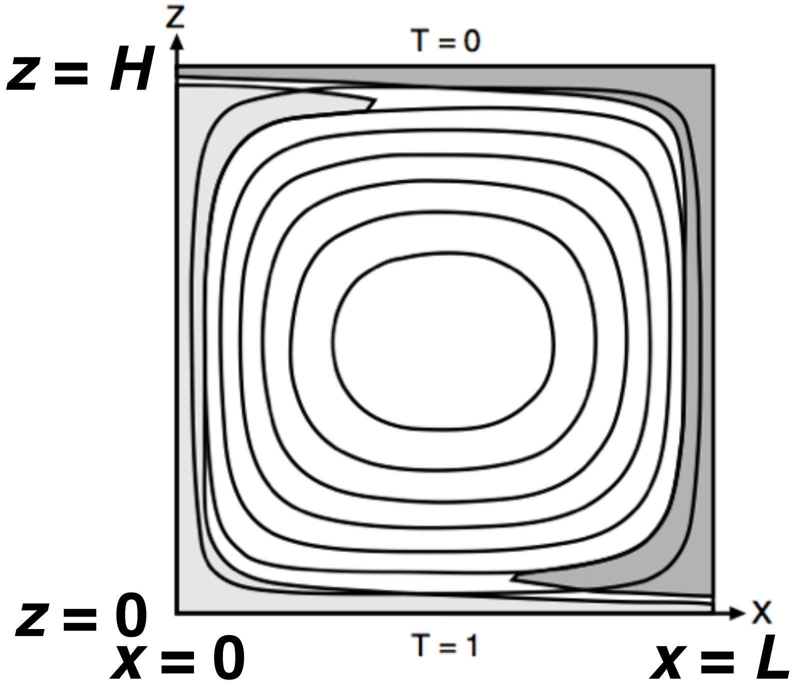


Figure 9.2. Contours of temperature for steady, two-dimensional, Rayleigh–Bénard convection in aspect ratio one cells heated from below (Jarvis, 1984), showing the development of thermal boundary layers with increasing Rayleigh number. Numbers indicate the ratio Ra/Ra_{cr}, with $Ra_{cr} = 779.27$.

Nusselt number (*Nu*)

- The Nusselt number (*Nu*) is a ratio of convective heat transfer to conductive heat transfer
- *Nu* is a *non-dimensional* number
- In the context of mantle dynamics, it's defined using horizontally averaged heat fluxes over the upper/lower boundaries



$$Nu = \frac{q_{\mathrm{surf.}}}{q_{\mathrm{cond.}}} = \frac{1}{L}\int_0^L \left(\frac{\partial T}{\partial z}\right)_{z=H} dx \Big/ \frac{\Delta T}{H}$$

(assumes constant conductivity, *k*)

- *Nu* provides a measure of the efficiency of heat loss through the surface via advection

Nu-Ra scaling laws

Problem dependence

(iso-viscous: moderate *Ra*)

$$Nu = cRa^{\beta}$$

$$c = 0.27, \quad \beta = 0.3185$$

(iso-viscous: high Ra)

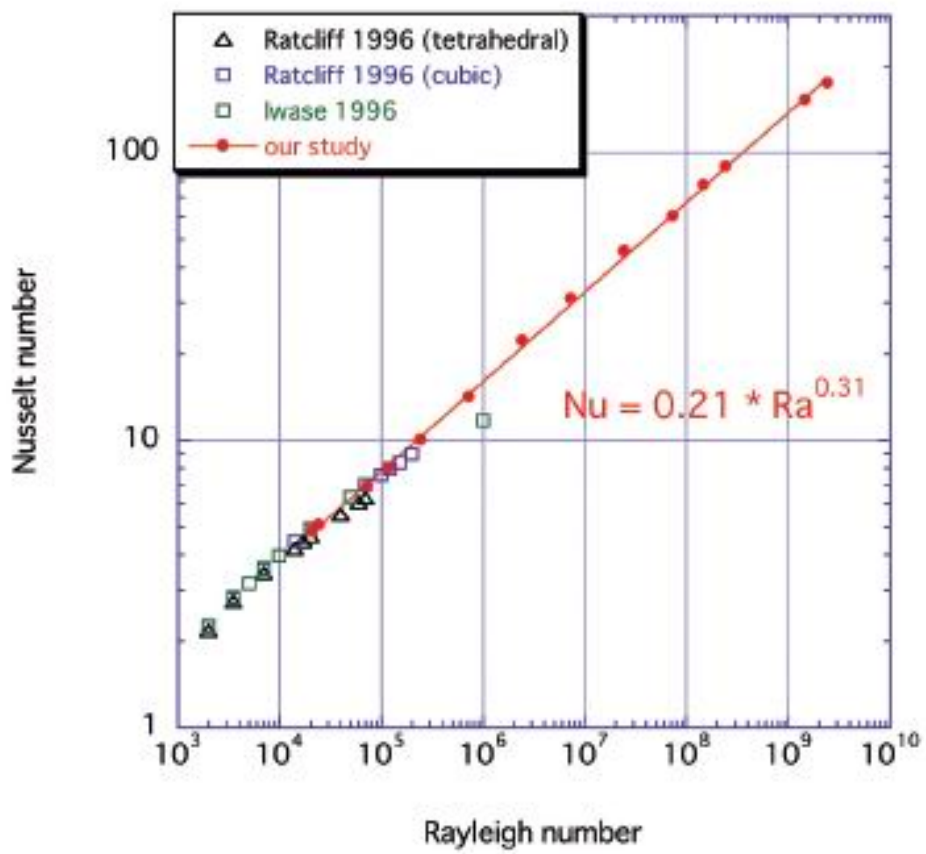


Fig. 5 Summary of the relationship between *Ra* and *Nu* for spherical shell with infinite *Pr* fluid. Our data (red points) suggests $Nu \sim Ra^{0.31}$ for wide range of *Ra*.

Fukao et al. (2008)

(variable viscosity)

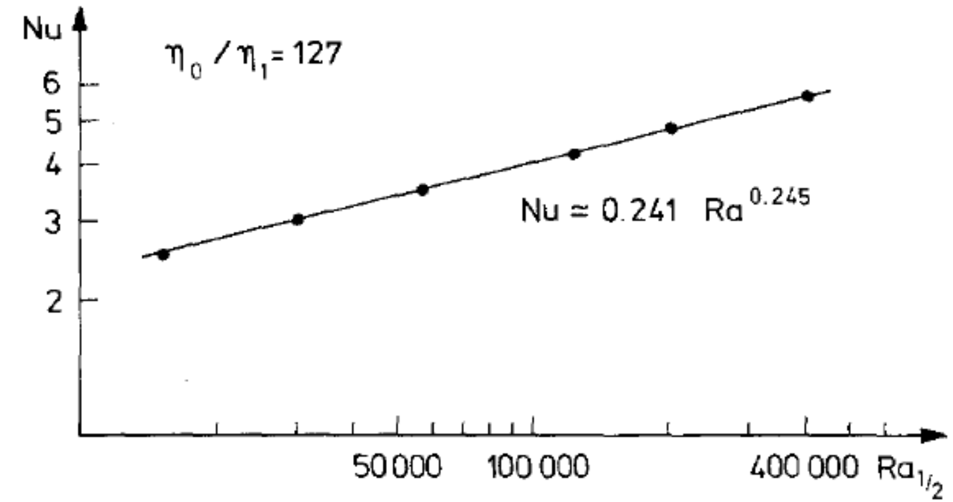


Fig. 2. Nusselt–Rayleigh number relationship for a fluid with rheological properties as used by Booker (1976) and a fixed viscosity ratio of 127. Upper and lower boundary are rigid.

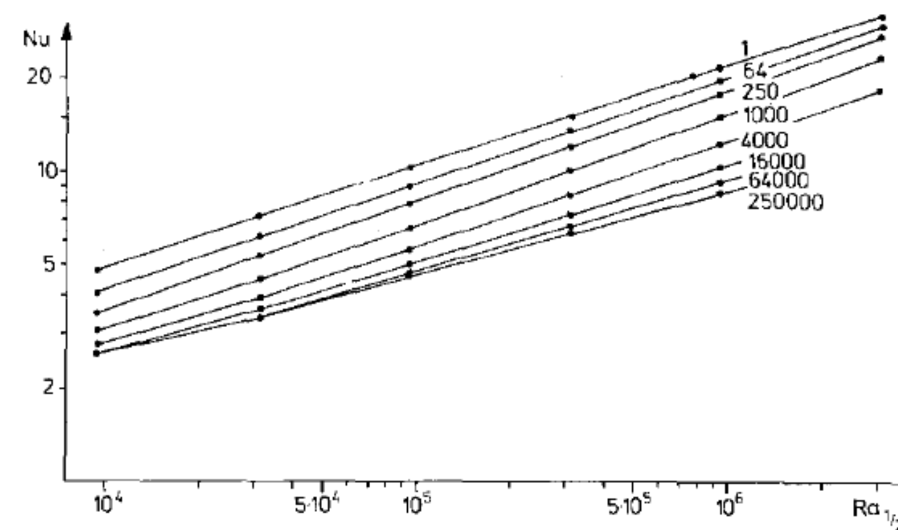
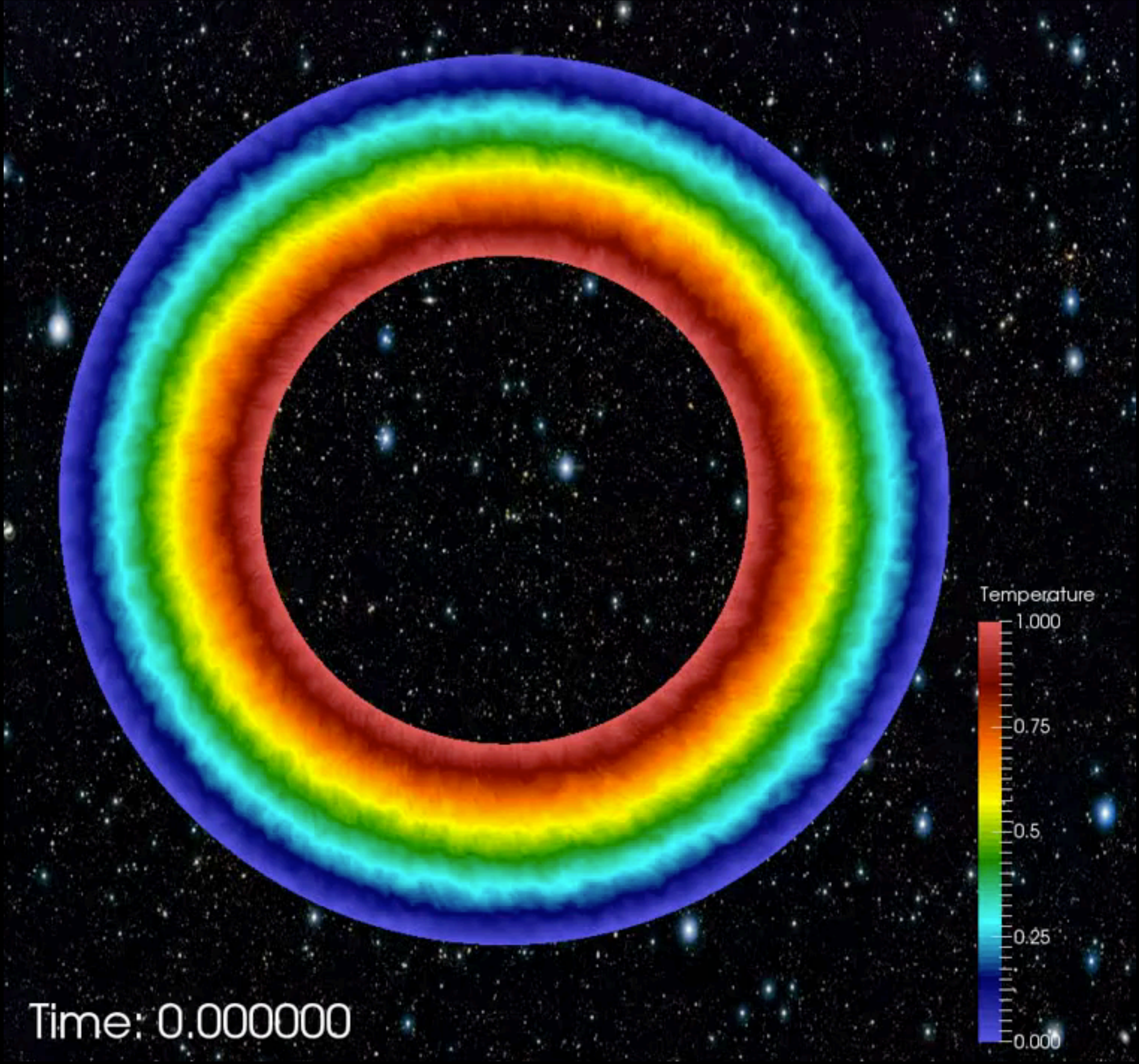


Fig. 7. Nusselt number versus Rayleigh number calculated with the viscosity at the mean of top and bottom temperature for variable viscosity convection with free boundaries. The numbers attached to the curves indicate the viscosity ratio

Christensen (1984)



https://youtu.be/t1UH9Qjy28I

Summary

- Introduced the fundamental equations which describe the long-time evolution of the mantle
  - conservation of momentum for highly viscous fluids
  - incompressibility
  - dynamic similarity
  - Boussinesq approximation
  - important non-dimensional numbers for mantle convection: Rayleigh number (*Ra*) and Nusselt number (*Nu*)