Derivation of the boundary layer theory

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Questions 1 and 2 will be corrected directly during the class. Only questions 3 and 4 should be done as a homework

We assume an equilibrium situation in a convecting domain heated from below and cooled from above. The surface temperature is T_s, the bottom temperature is T_b, and the boundary conditions are free slip. The Arrhenius viscosity η depends on temperature:

$$\eta_A(T) = \eta_0 \exp\left(\frac{E}{RT} - \frac{E}{RT_i}\right) \tag{1}$$

where T is the temperature, η_0 is a reference viscosity, E is an activation energy, R is the Boltzmann constant (8.314) and T_i is the internal temperature.

The Frank-Kamenetskii approximation

The Frank-Kamenetskii approximation is used to define a viscosity similar to the Arrhenius definition in the internal region but lower in the lithosphere (see figure 1). In the Frank-Kamenetskii approximation, in the internal region, the viscosity becomes:

$$\eta_{FK}(T) \simeq \eta_0 \exp\left(\theta(T_i - T)\right) \tag{2}$$

derivation of the Frank-Kamenetskii parameter

The Frank-Kamenetskii parameter θ represents the temperature dependence of the viscosity in the internal region.

- Question 1

In the internal region, one can consider that $\eta_{FK}(T) = \eta_A(T)$. Using

$$\frac{1}{x} - 1 \simeq 1 - x, \quad x \to 1, \tag{3}$$

derive the Frank-Kamenetskii parameter as a function of E, R and T_i.

- Question 2

When the viscosity is homogeneous (does not depend on temperature), the integral of the mechanical work in the whole domain is equal to the heat flow. Mechanical work is defined by the contraction of the deviatoric stress and strain rate tensors:

$$\Psi = \boldsymbol{\tau} : \dot{\boldsymbol{\epsilon}} \tag{4}$$

where the contraction of the tensors is the sum of all components. By simplicity, we will consider here only the average:

$$\Psi = \tau\dot{\epsilon}. \tag{5}$$

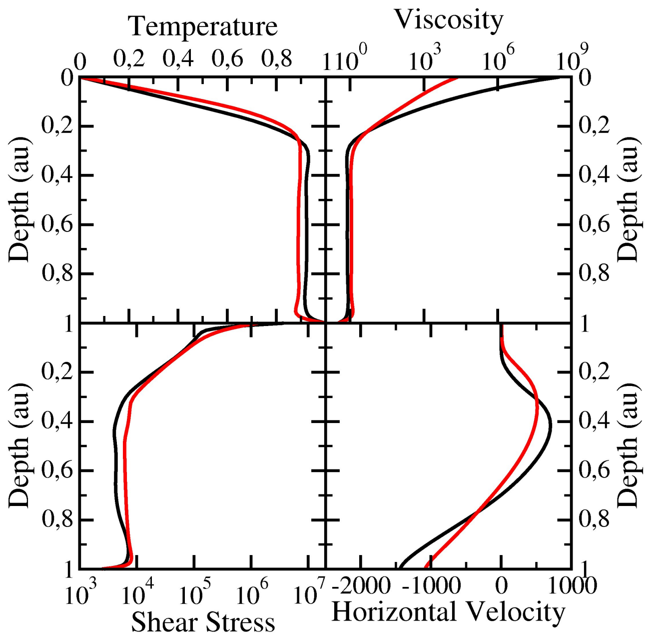

Figure 1: Red: Frank-Kamenetskii approximation. Black: Arrhenius. For similar equilibrium temperature profiles, the viscosity profiles are similar, except in the lithosphere.

Following dimensional analysis, we assume that the lithosphere thickness δ is related to the surface velocity v:

$$\delta = \left(\frac{\kappa h}{v}\right)^{1/2} \tag{6}$$

where κ is the thermal diffusivity, and h the thickness of the mantle.

Using the assumption mechanical work = surface heat flow in a cubic domain (in which the deformation is homogeneously distributed), find the lithosphere thickness as a function of the Rayleigh number. Note that dimensionaly:

$$\left[\frac{\alpha g h}{C_p}\right] = 1 \tag{7}$$

and

$$k = \kappa \rho C_p. \tag{8}$$

Reminder, the internal Rayleigh number is given by:

$$Ra_i = \frac{\alpha \rho g \Delta T h^3}{\kappa \eta_i} \tag{9}$$

- Question 3

Give the internal velocity as a function of the Rayleigh number.

- Question 4

We now consider the stagnant lid regime situation. The mantle deforms only below a frozen lithosphere. At the equilibrium, the internal temperature is very close to the core temperature. The temperature contrast inside the deforming domain is now ΔT_{rh} (where rh stands for "rheological"). In such case, in the internal region, the viscosity can be considered homogeneous. Therefore, the scaling laws previously derived still apply. The deforming boundary layers inside the deforming domain are a thickness δ_{rh} (they correspond to the lithosphere in the isoviscous case)

Using the continuity of heat flux at the base of the lithosphere:

$$\frac{\delta_{rh}}{\Delta T_{rh}} = \frac{\delta}{\Delta T} \tag{10}$$

where δ is the lithosphere thickness (the stagnant lid thickness).

We define a dimensionless Frank-Kamenetskii parameter p:

$$p = \theta \Delta T \tag{11}$$

The rheological temperature scale in the deforming domain is defined by dimensional analysis:

$$\Delta T_{rh} = \frac{\Delta T}{p} \tag{12}$$

Find the lithosphere thickness as a function of the dimensionless Frank-Kamenetskii parameter and the internal Rayleigh number.