Boundary Layer Theory in a 1D vertical profile

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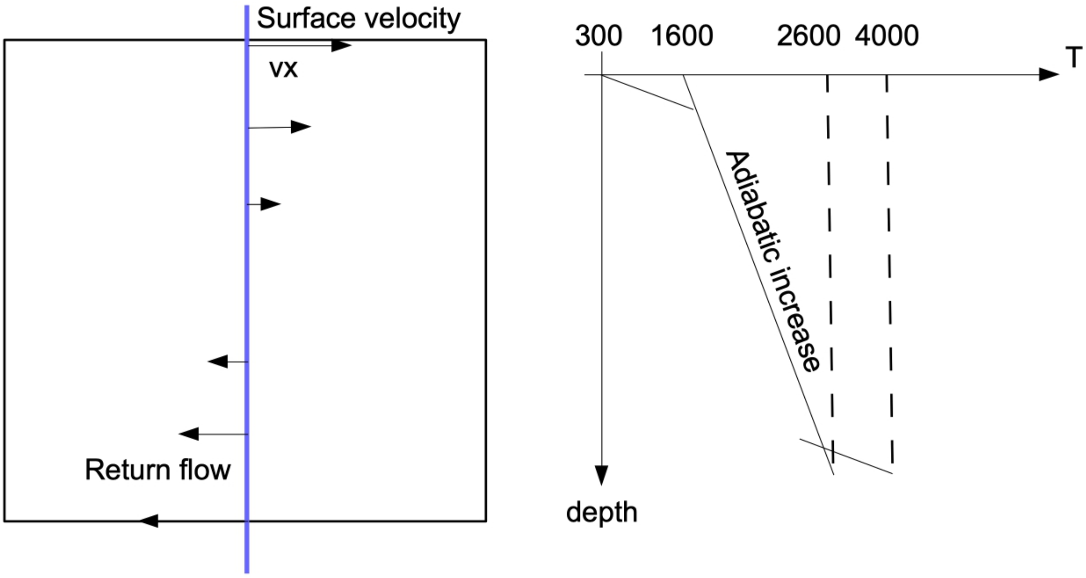

Figure 1: Geometry of the problem

Let's look into a vertical slice of a convection cell on Earth, assuming plate tectonics operates. Boundary layer theory would assume that the mechanical work is homogeneous in this profile. Since we look at the centre of the convection cell, we neglect upward and downward flow. Only lateral velocities are non zero.

If the mechanical work is homogeneous then one can use:

$$\Psi = \tau \cdot \dot{\epsilon} = 2\eta\dot{\epsilon}^2 = 2\eta \left(\frac{\partial v_x}{\partial z}\right)^2 \tag{1}$$

This implies

$$\Leftrightarrow \frac{\partial v_x}{\partial z} = \sqrt{\frac{\Psi}{2\eta}} \tag{2}$$

Using a surface velocity corresponding to the average velocity of Earth's oceanic plates ($\sim$ 3 cm/yr), one can calculate the lateral velocity profile by integrating from top to bottom.

$$v_x(z+\delta z) = v_x(z) + \frac{\partial v_x}{\partial z}\delta z \tag{3}$$

However, to ensure mass conservation in your convection cell, in cartesian geometry and in an incompressible model, one should target:

$$\int_{\text{top}}^{\text{bottom}} v_x dz = 0 \tag{4}$$

1 Homogeneous viscosity

- **1:** Use the heat flux ϕ of the ocean's floor on Earth (100 mW/m^2) to calculate Ψ. Remember that Pa·m^3/s=J/s=W. Use the thickness of the mantle (2890 km) and dimensional analysis. Remember that the fundamental assumption of boundary layer theory is:

$$\int \Psi dV = \int \phi dS \tag{5}$$

- **2:** Calculate a velocity profile using a reasonable reference viscosity.
- **3:** Check if the mass conservation equation is fulfilled.
- **4:** Perform an iterative search of η such that mass conservation is fulfilled. What do you get? The solution should be close to $4 \cdot 10^{22}$ Pa·s.

2 Lithosphere, upper and lower mantle viscosities

In this section, we consider that there are 3 viscosities accros the profile: lithosphere (10^{24} Pa·s), upper mantle (tbd) and lower mantle (tbd).

- **1:** Impose an upper mantle viscosity of 10^{20} Pa·s and a lower mantle viscosity of 10^{22} Pa·s.
- **2:** Using the value of Ψ estimated previously, calculate the velocity profile.
- **3:** Fixing the lower mantle viscosity, perform an iterative search of the upper mantle viscosity to fulfill mass conservation.
- **4:** Fixing the upper mantle viscosity, perform an iterative search of the lower mantle viscosity to fulfill mass conservation.
- **5:** Fixing a viscosity contrast of 30, perform an iterative search of the upper mantle viscosity to fulfill mass conservation. I find here and upper mantle viscosity of $8.17 \cdot 10^{21}$ Pa·s and a lower mantle viscosity of $2.45 \cdot 10^{23}$ Pa·s.

3 Depth-dependent viscosity

In this section, we use the Arrhenius equation to build a more detailed model of the mantle. We use the rheological formulation:

$$\dot{\epsilon} = A\left(\frac{\tau}{\mu}\right)\left(\frac{b}{d}\right)^m \exp\left(-\frac{E+PV}{RT}\right), \tag{6}$$

with $\mu = 80$ GPa, $b = 0.55$ nm, $R = 8.314$ J/K/mol, $m = 3$. The viscosity can be obtained using $\tau = 2\eta\dot{\epsilon}$, such that:

$$\begin{aligned}
\dot{\epsilon} &= A\left(\frac{2\eta\dot{\epsilon}}{\mu}\right)\left(\frac{b}{d}\right)^m \exp\left(-\frac{E+PV}{RT}\right) && (7)\\
\Leftrightarrow 1 &= A\left(\frac{2\eta}{\mu}\right)\left(\frac{b}{d}\right)^m \exp\left(-\frac{E+PV}{RT}\right) && (8)\\
\Leftrightarrow \eta &= \left[A\left(\frac{2}{\mu}\right)\left(\frac{b}{d}\right)^m \exp\left(-\frac{E+PV}{RT}\right)\right]^{-1} && (9)\\
\Leftrightarrow \eta &= \left(\frac{\mu}{2A}\right)\left(\frac{d}{b}\right)^m \exp\left(\frac{E+PV}{RT}\right) && (10)
\end{aligned}$$

- **1:** Since the Arrhenius equation strongly depends on temperature, you need to build a temperature profile of the mantle. Impose a lithosphere (100 km thick, i.e., by simplicity consider a temperature gradient of 13K per km) and an adiabatic temperature gradient with depth (1000K increase from surface to CMB). Use a potential surface temperature of 1600K. Impose a boundary layer at the base of the mantle. The core temperature is considered to be 4000K. For example, to determine the CMB boundary layer thickness, use the same temperature gradient than in the lithospere. Also build a pressure profile linear with depth (CMB pressure=130 GPa).
- **2:** Impose an activation Energy of 300 kJ/mol, no viscosity jump, no activation volume, a grain size of 1mm. Limit your lithosphere viscosity to a value of 10^{24} Pa·s. Look for a rheological prefactor which fulfills mass conservation.
- **3:** Use the rheological coefficients given in the class (for diffusion creep: $A = 8.7\,10^{15}$). Impose a viscosity jump of 30 at the 660. Consider a grain size of 1 mm. Find the activation volume which fulfills mass conservation.