Introduction to Finite Element Modelling in Geosciences:
2D Stokes Flow

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1 Overview

Many problems in Earth science involve the flow of fluids. One of the most important examples of which is the slow viscous deformation of rock. The purpose of this script is to present an introduction to the equations which govern the motion of very viscous fluid and to study how these equations can be solved using the finite element method.

The mechanics of a fluid is governed by four sets of equations, conservation of mass (continuity), conservation of momentum (force balance), a relationship between strain rate and velocity and a constitutive relationship. The force balance in two dimensions is governed by the equations

$$\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} &= 0 \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} &= \rho g, \end{aligned} \tag{1}$$

where σ_{ij} is the stress tensor, ρ is density and g is gravitational acceleration (note inertial terms are ignored). The equation for the conservation of mass of an incompressible fluid is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2}$$

where u and v are velocities in x and y direction respectively. The constitutive relationship for an incompressible, Newtonian viscous material is

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = -p \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2\eta & 0 & 0 \\ 0 & 2\eta & 0 \\ 0 & 0 & 2\eta \end{bmatrix} \begin{bmatrix} \dot{\epsilon}_{xx} \\ \dot{\epsilon}_{yy} \\ \dot{\epsilon}_{xy} \end{bmatrix}, \tag{3}$$

where η is the viscosity, $\dot{\epsilon}_{ij}$ are strain rates and p is the pressure. Finally, the kinematic relationship between strain rates and velocities is defined as

$$\begin{bmatrix} \dot{\epsilon}_{xx} \\ \dot{\epsilon}_{yy} \\ \dot{\epsilon}_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \end{bmatrix}. \tag{4}$$

When written out in full, these equations in 2D have the form

$$
\begin{aligned}
-\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(2\eta\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\eta\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right) &= 0 && (5)\\
-\frac{\partial p}{\partial y} + \frac{\partial}{\partial y}\left(2\eta\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial x}\left(\eta\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right) &= \rho g && (6)\\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0. && (7)
\end{aligned}
$$

This set of coupled partial differential equations are commonly known as the Stokes equations. Using matrix notation, Eqs. (1), (2), (3) & (4) can be written compactly as

$$\mathbf{B}^T\hat{\sigma} = \mathbf{f} \tag{8}$$
$$\mathbf{m}^T\mathbf{B}\mathbf{e} = 0 \tag{9}$$
$$\hat{\sigma} = -p\mathbf{m} + \mathbf{D}\dot{\mathbf{e}} \tag{10}$$
$$\dot{\mathbf{e}} = \mathbf{B}\mathbf{e}, \tag{11}$$

where

$$\mathbf{e} = \begin{bmatrix} u \\ v \end{bmatrix}, \qquad \mathbf{f} = \begin{bmatrix} 0 \\ \rho g \end{bmatrix}, \tag{12}$$

$$\mathbf{m} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad \nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix}, \tag{13}$$

$$\mathbf{B} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}, \tag{14}$$

$$\mathbf{D} = \begin{bmatrix} 2\eta & 0 & 0 \\ 0 & 2\eta & 0 \\ 0 & 0 & \eta \end{bmatrix}, \tag{15}$$

and

$$\dot{\mathbf{e}} = \begin{bmatrix} \dot{\epsilon}_{xx} \\ \dot{\epsilon}_{yy} \\ \dot{\gamma}_{xy} \end{bmatrix}, \qquad \hat{\sigma} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix}. \tag{16}$$

Note in the above equation that $\dot{\gamma}_{xy} = 2\dot{\epsilon}_{xy}$. The reader should verify that Eqs. (8) - (10) are correct. In the Stokes formulation considered here, one eliminates $\hat{\sigma}$ and $\dot{\mathbf{e}}$ in the following manner

$$\mathbf{B}^T\hat{\sigma} = \mathbf{f} \tag{17}$$
$$\mathbf{B}^T\mathbf{D}\dot{\mathbf{e}} - p\mathbf{B}^T\mathbf{m} = \mathbf{f} \tag{18}$$
$$\mathbf{B}^T\mathbf{D}\mathbf{B}\mathbf{e} - p\mathbf{B}^T\mathbf{m} = \mathbf{f}, \tag{19}$$

to obtain an expression only in terms of the velocity $\mathbf{u}$ and pressure p. Thus, the governing equations are

$$\mathbf{B}^T\mathbf{D}\mathbf{B}\mathbf{e} - \nabla p = \mathbf{f} \tag{20}$$

and

$$\mathbf{m}^T\mathbf{B}\mathbf{e} = 0, \tag{21}$$

which is a set of three equations (remember that Equation (20) consists of two equations), for the three unknowns u, v and p.

2 FE discretisation

One simple and (barely) adequate method for discretising Equations (5), (6) and (7) with quadrilateral elements is to use the 9 node biquadratic shape functions for velocities and the 4 node bilinear shape function for the pressure. This configuration is depicted in Figure 1. Note that the same quadrilateral geometry is used to define the approximation for both velocity and pressure. We also note that the corners of the velocity element coincide with the corners of the pressure element. The two velocities are approximated as

$$u(x,y) \approx \begin{bmatrix} N_1 & N_2 & N_3 & N_4 & N_5 & N_6 & N_7 & N_8 & N_9 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{bmatrix} = \mathbf{N}\mathbf{u}^e, \tag{22}$$

$$v(x,y) \approx \begin{bmatrix} N_1 & N_2 & N_3 & N_4 & N_5 & N_6 & N_7 & N_8 & N_9 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \end{bmatrix} = \mathbf{N}\mathbf{v}^e, \tag{23}$$

or

$$\begin{bmatrix} u \\ v \end{bmatrix} \approx \begin{bmatrix} \mathbf{N} & \mathbf{0} \\ \mathbf{0} & \mathbf{N} \end{bmatrix} \begin{bmatrix} \mathbf{u}^e \\ \mathbf{v}^e \end{bmatrix} = \hat{\mathbf{N}}\mathbf{U}^e. \tag{24}$$

The pressure is approximated via

$$p(x,y) \approx \begin{bmatrix} \bar{N}_1 & \bar{N}_2 & \bar{N}_3 & \bar{N}_4 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \mathbf{N}_p\mathbf{p}^e. \tag{25}$$

Note in the above that we denote the basis functions for the velocity via N_i and those for pressure via $\bar{N}_i$.

First we consider the weak form of the momentum equations. To simplify the derivation, we multiply the stress gradient (in Voigt notation) by the velocity

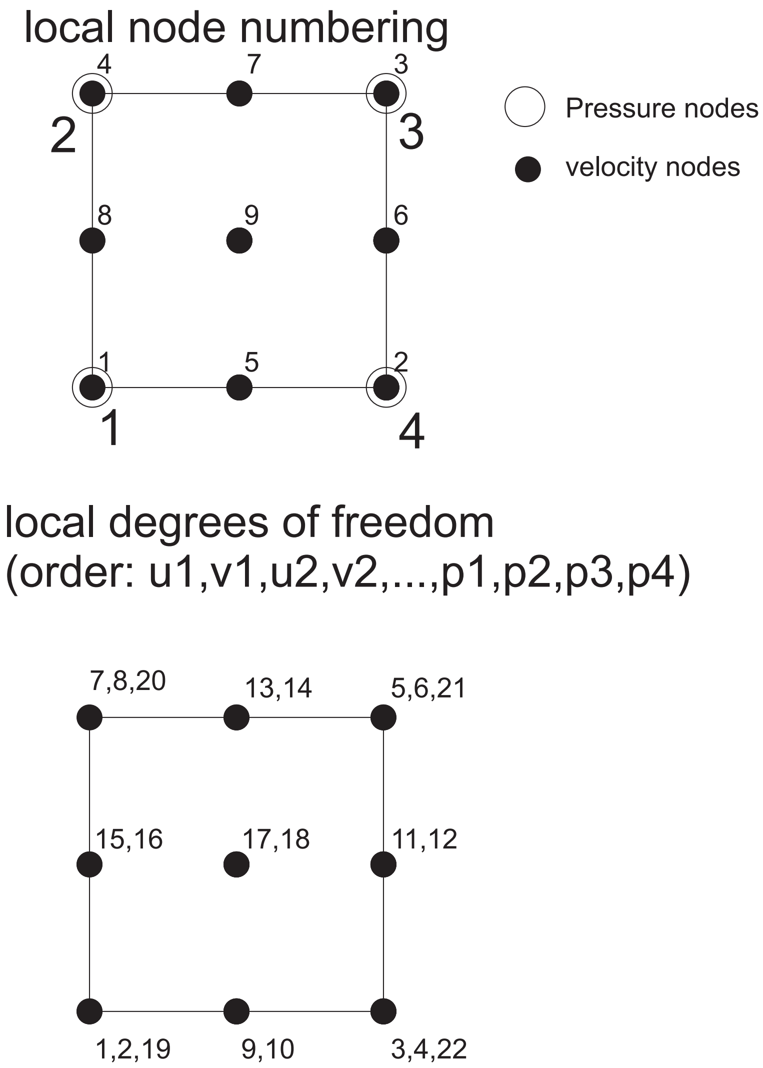

Figure 1: A single finite element illustrating the Q2-Q1 velocity-pressure formulation used to solve Stokes flow. A biquadratic shape function (Q_2) is used for velocity and a bilinear shape function (Q_1) for pressure. The nodes associated with velocity (solid circles) possess two degrees of freedom (for u, v) and the nodes associated with the pressure (open circles) possess one degree of freedom.

basis functions for u and v and integrate over the element volume Ω^e. In 2D this yields

$$\int_{\Omega^e} \begin{bmatrix} \mathbf{N}^T & 0 \\ 0 & \mathbf{N}^T \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} dV = \int_{\Omega^e} \begin{bmatrix} \mathbf{N}^T & 0 \\ 0 & \mathbf{N}^T \end{bmatrix} \begin{bmatrix} f_x \\ f_y \end{bmatrix} dV. \tag{26}$$

Then applying integration by parts yields

$$\begin{aligned} \int_{\Omega^e} \begin{bmatrix} \frac{\partial \mathbf{N}^T}{\partial x} & 0 & \frac{\partial \mathbf{N}^T}{\partial y} \\ 0 & \frac{\partial \mathbf{N}^T}{\partial y} & \frac{\partial \mathbf{N}^T}{\partial x} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} dV = \\ \int_{\Omega^e} \begin{bmatrix} \mathbf{N}^T & 0 \\ 0 & \mathbf{N}^T \end{bmatrix} \begin{bmatrix} f_x \\ f_y \end{bmatrix} dV - \oint_{\partial\Omega^e} \begin{bmatrix} \mathbf{N}^T(\sigma_{xx}n_x + \sigma_{xy}n_y) \\ \mathbf{N}^T(\sigma_{yx}n_x + \sigma_{yy}n_y) \end{bmatrix} dS. \end{aligned} \tag{27}$$

Denoting the Neumann boundary conditions as $t_i = \sigma_{ij}n_j$ and the boundary segment of each element as Γ^e, we can simplify the weak form above to

$$\int_{\Omega^e} \begin{bmatrix} \frac{\partial \mathbf{N}^T}{\partial x} & 0 & \frac{\partial \mathbf{N}^T}{\partial y} \\ 0 & \frac{\partial \mathbf{N}^T}{\partial y} & \frac{\partial \mathbf{N}^T}{\partial x} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} dV = \int_{\Omega^e} \begin{bmatrix} \mathbf{N}^T f_x \\ \mathbf{N}^T f_y \end{bmatrix} dV - \oint_{\Gamma^e} \begin{bmatrix} \mathbf{N}^T t_x \\ \mathbf{N}^T t_y \end{bmatrix} dS. \tag{28}$$

To complete the derivation, we introduce the discrete strain rate operator

$$\hat{\mathbf{B}} = \mathbf{B}\hat{\mathbf{N}}, \tag{29}$$

and then insert the definition of the discrete stress

$$\hat{\sigma} = \mathbf{D}\hat{\mathbf{B}}\mathbf{U}^e - \mathbf{m}\mathbf{N}_p\mathbf{p}^e,$$

to give

$$\begin{aligned} \int_{\Omega^e} \begin{bmatrix} \frac{\partial \mathbf{N}^T}{\partial x} & 0 & \frac{\partial \mathbf{N}^T}{\partial y} \\ 0 & \frac{\partial \mathbf{N}^T}{\partial y} & \frac{\partial \mathbf{N}^T}{\partial x} \end{bmatrix} \left(\mathbf{D}\hat{\mathbf{B}}\mathbf{U}^e - \mathbf{m}\mathbf{N}_p\mathbf{p}^e \right) dV = \\ \int_{\Omega^e} \begin{bmatrix} \mathbf{N}^T f_x \\ \mathbf{N}^T f_y \end{bmatrix} dV - \oint_{\Gamma^e} \begin{bmatrix} \mathbf{N}^T t_x \\ \mathbf{N}^T t_y \end{bmatrix} dS, \end{aligned} \tag{30}$$

or

$$\mathbf{K}^e\mathbf{U}^e + \mathbf{G}^e\mathbf{p}^e = \mathbf{f}^e, \tag{31}$$

in which

$$\mathbf{K}^e = \int_{\Omega^e} \hat{\mathbf{B}}^T\mathbf{D}\hat{\mathbf{B}}\, dV, \tag{32}$$

$$\mathbf{G}^e = -\int_{\Omega^e} \hat{\mathbf{B}}^T\mathbf{m}\mathbf{N}_p\, dV \tag{33}$$

and

$$\mathbf{f}^e = \int_{\Omega^e} \begin{bmatrix} \mathbf{N}^T f_x \\ \mathbf{N}^T f_y \end{bmatrix} dV - \oint_{\Gamma^e} \begin{bmatrix} \mathbf{N}^T t_x \\ \mathbf{N}^T t_y \end{bmatrix} dS. \tag{34}$$

In comparison, the weak form of the continuity equation is much simpler to derive. All that is required is to multiply the continuity equation by $\mathbf{N}_p^T$ and integrate over the element volume to yield

$$-\int_{\Omega} \mathbf{N}_p^T\mathbf{m}^T\hat{\mathbf{B}}\mathbf{U}^e\, dV = \mathbf{0}, \tag{35}$$

or

$$(\mathbf{G}^e)^T \mathbf{U}^e = \mathbf{0}. \tag{36}$$

Note that we multiplied the weak continuity equation by minus one simply to make formulation symmetric (The reader should verify the system is symmetric). This doesn't change the system of equations as the right side of the continuity equations is the zero vector. The complete discretised Stokes problem (on the element level) can be compactly expressed via

$$\begin{bmatrix} \mathbf{K}^e & \mathbf{G}^e \\ (\mathbf{G}^e)^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{U}^e \\ \mathbf{p}^e \end{bmatrix} = \begin{bmatrix} \mathbf{f}^e \\ \mathbf{0} \end{bmatrix}. \tag{37}$$

One should note that the element matrices all have different dimensions. Using the 9-noded biquadratic elements for velocity, there are 18 velocity unknowns per element. The bilinear pressure element results in 4 pressure degrees of freedom per element. Consequently, $\mathbf{K}^e$ has dimensions 18×18 and $\mathbf{G}^e$ has dimensions 18×4. Care should be taken in carrying out the matrix-vector products as the discrete velocity and pressure vectors are of different length. All the element matrices defined above can be integrated and assembled in the standard finite element fashion.

3 Exercises

1. Write a FEM code to solve the Stokes equations for the unknowns u, v and p using boundary conditions and the Rayleigh-Taylor setup shown in Figure 2. Use quadratic shape functions for velocity and continuous linear shape functions for pressure. Free slip conditions are implemented by only setting the boundary-normal velocities to zero (so e.g., $V_x = 0$ on the left and right boundary). Model time evolution by advecting the grid, and also study what happens if the upper boundary is a free surface (set no boundary conditions in this case).

2. Plot the second invariant of the strainrate tensor ($\dot{\epsilon}_{II} = \frac{1}{2}(\dot{\epsilon}_{xx}^2 + \dot{\epsilon}_{yy}^2 + 2\epsilon_{xy}^2)^{0.5}$) for the previous model.

3. Create a code to simulate viscous detachment folding subjected to a background pure shear deformation. Ask the instructors for hints.

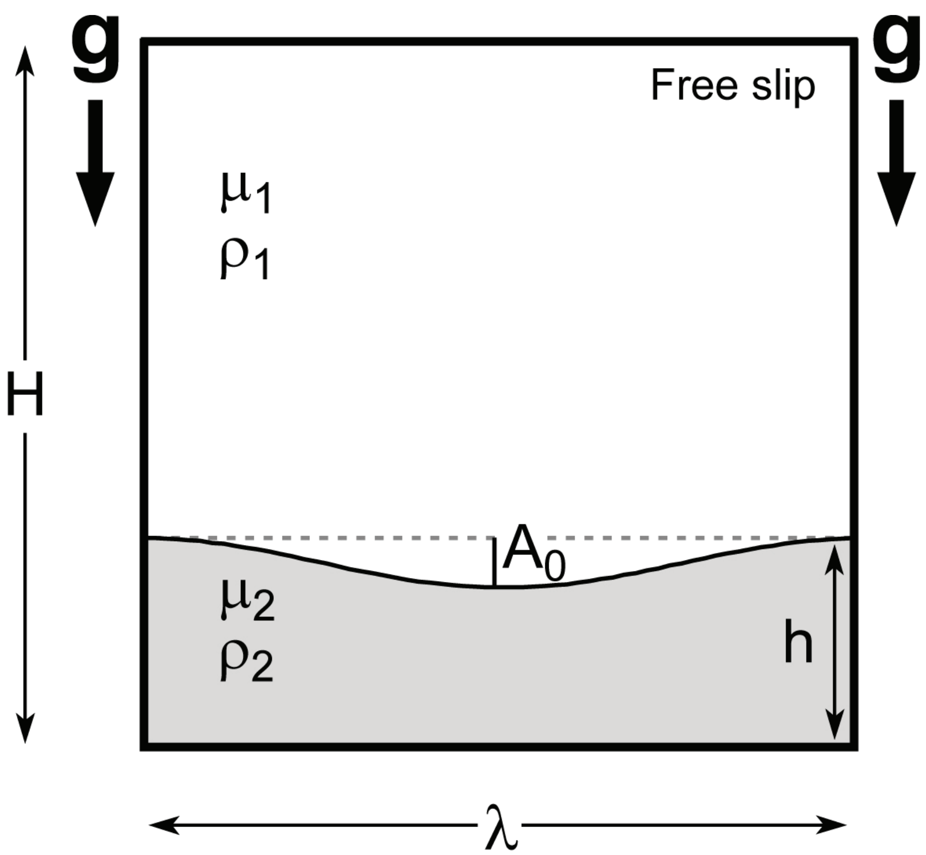

Figure 2: Rayleigh Taylor setup in which a fluid of higher density is superposed on top of a fluid with lower density and the interface between the two fluids is perturbed in a sinusoidal manner. Employ $\lambda = H = 1$, $h = 0.5, A_0 = 10^{-2}, g = 1, \mu_1 = 1, \mu_2 = 1, \rho_1 = 1, \rho_2 = 0$.