Introduction to Finite Element Modelling in Geosciences: **Formula Sheet**

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Vector operations

$$\mathrm{grad}(u) = \nabla u = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} u \tag{1}$$

$$\mathrm{div}(\mathbf{v}) = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix} \cdot \mathbf{v} = \nabla^T \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} \tag{2}$$

Integration by parts (1D)

$$\int_a^b f(x) \frac{dg(x)}{dx}\, dx = -\int_a^b \frac{df(x)}{dx} g(x)\, dx + \left[f(x)g(x)\right]_a^b \tag{3}$$

Integration by parts (higher dimensions)

$$\int_\Omega f \frac{\partial g}{\partial x_i}\, dV = -\int_\Omega \frac{\partial f}{\partial x_i} g\, dV + \oint_{\partial\Omega} f g\, n_i\, dS \tag{4}$$

$$\int_\Omega \boldsymbol{f} \cdot \nabla g\, dV = -\int_\Omega g \nabla \cdot \boldsymbol{f}\, dV + \oint_{\partial\Omega} g \boldsymbol{f} \cdot \boldsymbol{n}\, dS \tag{5}$$

$$\int_\Omega \nabla f \cdot \nabla g\, dV = -\int_\Omega f \nabla^2 g\, dV + \oint_{\partial\Omega} f \nabla g \cdot \boldsymbol{n}\, dS \tag{6}$$

Tensor products

Given a vector of 1D functions, $P(s) = (P_1(s), P_2(s), \ldots, P_m(s))$, we can define a 2D tensor product with the coordinates (ξ, η)

$$\begin{aligned} N_{11}(\xi,\eta) &= P_1(\xi)P_1(\eta) \\ N_{21}(\xi,\eta) &= P_2(\xi)P_1(\eta) \\ N_{31}(\xi,\eta) &= P_3(\xi)P_1(\eta) \\ &\vdots \\ N_{m1}(\xi,\eta) &= P_m(\xi)P_1(\eta) \\ N_{12}(\xi,\eta) &= P_1(\xi)P_2(\eta) \\ &\vdots \\ N_{m2}(\xi,\eta) &= P_m(\xi)P_2(\eta) \\ &\vdots \\ N_{mm}(\xi,\eta) &= P_m(\xi)P_m(\eta) \end{aligned} \tag{7}$$

In general we have,

$$N_{ij} = P_i(\xi)P_j(\eta), \qquad \text{for all } 1 \leq i, j \leq m \tag{8}$$

A 3D tensor product can similarily be defined with the coordinates (ξ, η, ζ) via

$$N_{ijk} = P_i(\xi)P_j(\eta)P_k(\zeta), \qquad \text{for all } 1 \leq i, j, k \leq m \tag{9}$$

We note that the 1D basis used need not be the same length, for example one could define

$$N_{ijk} = U_i(\xi)V_j(\eta)W_k(\zeta), \tag{10}$$

where $U(s) = (U_1(s), \ldots, U_m(s))$, $V(s) = (V_1(s), \ldots, V_n(s))$, $W(s) = (W_1(s), \ldots, W_p(s))$ and where $m \neq n \neq p$.