Introduction to Finite Element Modelling in Geosciences: The Weak Form

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July 8, 2014

1 A prototype PDE

Consider the Poisson equation

$$\nabla^2 u + f = 0. \tag{1}$$

In order to obtain a unique solution of Eq (1) in some domain Ω, we require the prescription of the boundary conditions. We will denote the boundary of Ω via $\partial\Omega$ and the interior of the domain as $\bar{\Omega} = \Omega \setminus \partial\Omega$. Dirichlet boundary conditions specific the value of u along some region of the boundary which we denote by $\partial\Omega_D$. A Neumann boundary specifies the value of ∇u along some boundary segment $\partial\Omega_N$. At every point in space $\mathbf{x}$ along the entire boundary of Ω, a boundary condition must be specified. That is, $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$ and these segments do not overlap, i.e. $\partial\Omega_D \cap \partial\Omega_N = \emptyset$. Formally we can state this problem as:
Find u such that

$$\nabla^2 u + f = 0 \qquad \text{in } \bar{\Omega} \tag{2}$$

subject to

$$u = g_D \quad \text{on } \partial\Omega_D \qquad \text{and} \qquad \nabla u \cdot \mathbf{n} = g_N \quad \text{on } \partial\Omega_N, \tag{3}$$

where $\mathbf{n}$ is the outward pointing normal to the boundary $\partial\Omega$.

2 The weak form

To construct the weak form we introduce a *test function* v and we will require that

$$\int_\Omega \left(\nabla^2 u + f\right) v \, dV = 0. \tag{4}$$

Using Green's theorem, the above can be equivalently stated as

$$\int_\Omega \nabla u \cdot \nabla v \, dV = \int_\Omega v f \, dV + \oint_{\partial\Omega} v \frac{\partial u}{\partial n} \, dS. \tag{5}$$

We now discuss some details related to the weak form defined in Eq. (5). From Eq. (4) it is apparent that any solution u which satisfies Eq. (1) satisfies Eq. (4) for *any* choice of v. To consider whether the converse is true, we first observe that the required smoothness of u has been reduced, that is the derivative operating on u has changed from second order to first order. Thus Eq. (5) may have a solution say u', but u' may not be smooth enough to also be a solution of Eq. (1). For this reason, solutions of Eq. (5) are referred to as weak solutions. Given their weak nature, many possible weak solutions will exist which satisfy Eq. (5). We will discuss the class of functions from which weak solutions live within in the following paragraphs.

As of yet we have not discussed what types of functions are valid choices for u and v. One restriction on selecting v comes from the observation that the weak form in Eq. (5) contains a flux term $\frac{\partial u}{\partial n} = \nabla u \cdot \mathbf{n}$, defined over the entire boundary. Recalling the problem statement required to obtain a solution to the PDE (see Eq. (3)), we observe that the flux term in the weak form exactly matches the Neumann boundary condition, except that that Neumann condition is only enforced over the boundary segment $\partial\Omega_N$ and not over all $\partial\Omega$. Therefore, to enforce that the weak form is consistent with the boundary conditions of the PDE we require that $v = 0$ along the Dirichlet boundary $\partial\Omega_D$. Thus, for solutions of the PDE which only possess Dirichlet boundaries, i.e. $\Omega_N = \emptyset$, the surface integral in the weak form vanishes as v is constructed to be identically zero along this boundary region. The Dirichlet boundary also provides a restriction on the choice of u, name that any function used to define the weak solution must satisfy $u = g_D$ on $\partial\Omega_D$. Using this restrictions on u, v, we can state the weak form problem as:
Find u such that

$$\int_\Omega \nabla u \cdot \nabla v \, dV = \int_\Omega v f \, dV + \oint_{\partial\Omega_N} v \, g_N \, dS \qquad \text{in } \bar{\Omega}. \tag{6}$$

Another property to consider in choosing a valid v, and in defining the class of permissible functions within which weak solutions to Eq. (6) live within, come from the consideration of *function smoothness.* To define smoothness, we use an L_2 measure defined as

$$||u||_2 := \left(\int_\Omega u^2 \, dV \right)^{1/2}. \tag{7}$$

Any function u which satisfies

$$||u||_2 < \infty$$

is said to live within the space (i.e. the set of all functions) of L_2 functions. Considering the left hand side of Eq. (6) we observed that the equation is well-defined (doesn't blow up) if all the derivatives of u and v are in L_2. Accordingly, if this true and both f and g_N also live within L_2, then the right hand side of Eq. (6) will also be well bounded.

3 Discrete weak form

We will assume that the problem domain Ω, has been partitioned into Me elements and Nn nodes. One each node, i we will define a test function $v(\mathbf{x}) =$

$\hat{\phi}_i(\mathbf{x})$. Inserting this expression into Eq. (6) yields

$$\sum_i^{Nn} \int_\Omega \nabla u \cdot \nabla \hat{\phi}_i \, dV = \sum_i^{Nn} \int_\Omega \hat{\phi}_i f \, dV + \oint_{\partial\Omega_N} \hat{\phi}_i \, g_N \, dS. \tag{8}$$

The discrete solution space is then defined via

$$u(\mathbf{x}) = \phi_1(\mathbf{x}) u_1 + \phi_2(\mathbf{x}) u_2 + \cdots = \sum_{i=1}^{Nn} \phi_i(\mathbf{x}) u_i,$$

where each u_i represents the approximate to u at the node i. Upon substitution we have

$$\sum_i^{Nn} \sum_j^{Nn} \int_\Omega \nabla \hat{\phi}_i \cdot \nabla \phi_j u_j \, dV = \sum_i^{Nn} \int_\Omega \hat{\phi}_i f \, dV + \oint_{\partial\Omega_N} \hat{\phi}_i \, g_N \, dS. \tag{9}$$

By making the choice that the trial functions should be identical to the discrete solution space, that is choosing $\hat{\phi}_i = \phi_i$, we obtain the Galerkin approximation,

$$\sum_i^{Nn} \sum_j^{Nn} \int_\Omega \nabla \phi_i \cdot \nabla \phi_j u_j \, dV = \sum_i^{Nn} \int_\Omega \phi_i f \, dV + \oint_{\partial\Omega_N} \phi_i \, g_N \, dS. \tag{10}$$

The equation above defines a system of linear equation which can be more compactly expressed via a matrix-vector form.

4 Further reading

- For a discussion about the weak form, see sections 1.1 to 1.3 of Elman, Silvester & Wathan.
- For a discussion about evaluating the discrete weak form, see sections 1.4 of Elman, Silvester & Wathan.
- For a discussion on the same concepts with, read and work through pages 1-13 of Hughes.