Finite Element concept review

The FE procedure

- Take your favourite partial differential equation

  - Derive the weak form (e.g. multiply by test function then integrate)

  - Mesh your domain

  - Choose an interpolation function (*)

  - Define element stiffness matrices and element vectors (**)

  - Assemble element matrices/vectors into global stiffness matrix/vector

  - Solve

  - Post process... aka. make a beautiful image of your FE result

From 1D to 2D (*)

- Isoparametric elements
  - Shape functions are expressed in local coordinates (ξ, η)
  - Coordinate transformation (Jacobian) is required to define integral

(I) $\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \qquad J_{ij} = \frac{\partial x_j}{\partial \xi_i}$

(II) $\det(\mathbf{J}) = J_{11}J_{22} - J_{12}J_{21} \qquad \int_\Omega f(x,y)\, dxdy = \int_\square f(\xi,\eta) \det(\mathbf{J}) d\xi d\eta$

  - Derivatives w.r.t. (x,y) have to be expressed in terms of (ξ, η)

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} = \mathbf{J}^{-1} \begin{bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{bmatrix}$$

From 1D to 2D (**)

- Numerical quadrature
  - Multiple dimensional shape function
  - Variable coefficients (diffusivity)
  - Jacobian

These factors make the integrals more complex.

Analytic integration is not a practical option.

- Tabulated rules are used which define the location where to evaluate the function *f()* we wish to integrate, and *the weight* to associate with this term:

$$\int_\Omega f(\xi)\, dV \approx \sum_{i=1}^{n} w_i f(\xi_i) \det(J(\xi_i))$$

2D diffusion

- Partial differential equation $\text{in} \quad \Omega$

[case 1 : constant kappa]

$$\kappa \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = f$$

[case 2 : spatially varying, isotropic kappa]

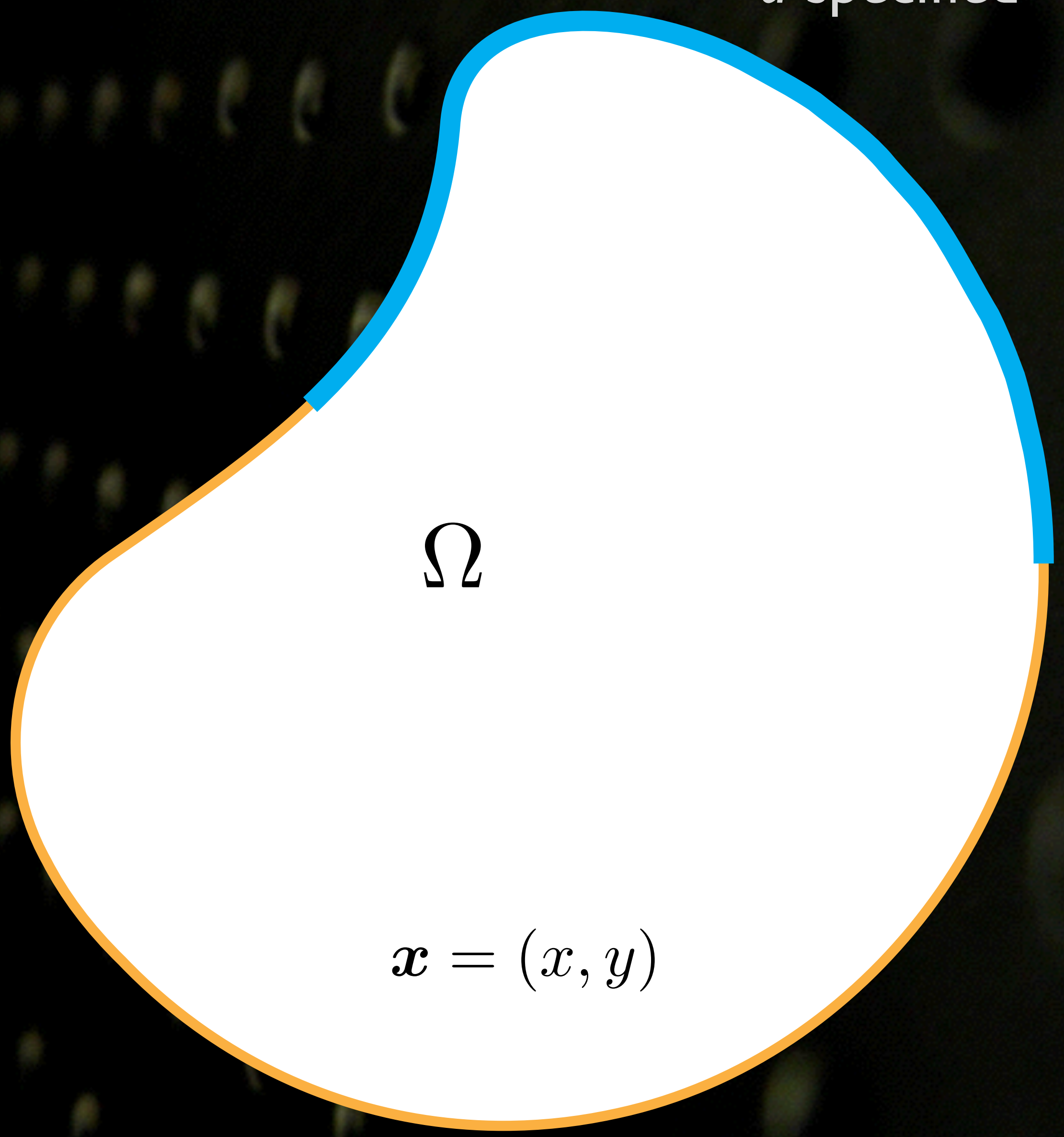
$$\frac{\partial}{\partial x} \left(\kappa(x,y) \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(\kappa(x,y) \frac{\partial u}{\partial y} \right) = f$$

- Boundary conditions

  - define the value of *u* (Dirichlet)

  - define the normal flux
    `div[k grad[u],n]` (Neumann)

Dirichlet:
u specified



Neumann:
`div[k grad[u],n]` specified

2D diffusion

- Partial differential equation $\text{in} \quad \Omega$

[case 2 : spatially varying, isotropic kappa]

$$\frac{\partial}{\partial x}\left(\kappa(x,y)\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\kappa(x,y)\frac{\partial u}{\partial y}\right)=f$$

[case 3 : spatially varying, anisotropic kappa]

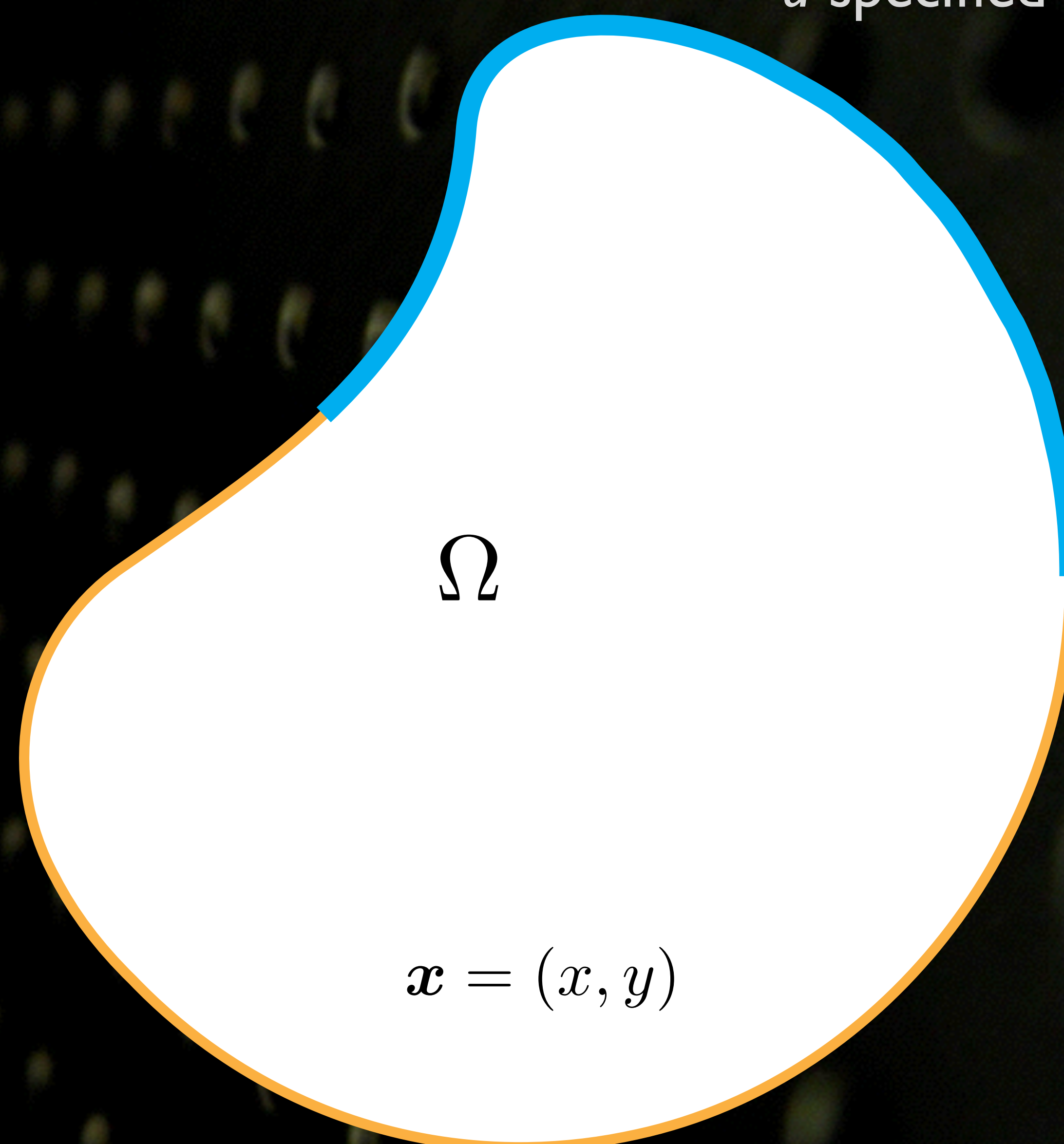
$$\frac{\partial}{\partial x_i}\left(\kappa_{ij}\frac{\partial}{\partial x_j}\right)u=f$$

- Boundary conditions

  - define the value of *u* (Dirichlet)

  - define the normal flux
    `div[k grad[`*u*`],`*n*`]` (Neumann)

Dirichlet:
u specified



Neumann:
`div[k grad[`*u*`],`*n*`]` specified