ETH
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Introduction to the FE method in geosciences

Lecture 3.2:

Isoparametric elements

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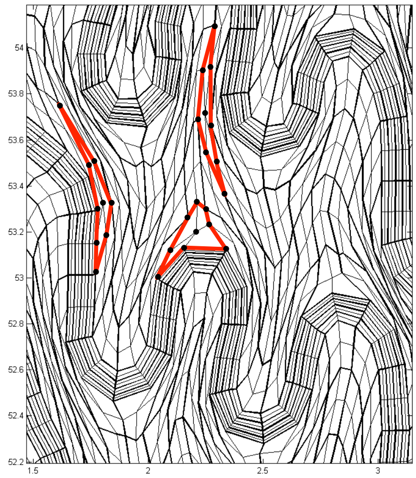
Motivation

→ Gauss-Legendre-Quadrature

- Numerical integration with Gauss-Legendre-Quadrature only works on an idealized Element
  - For x = -1 to 1 in 1D
  - For x = -1 to 1 and y = -1 to 1 in 2D
- So, it does not solve the problem of the distorted elements, yet.
- A coordinate transformation from the distorted element to the idealized element is needed in addition.

Distorted vs. idealized element



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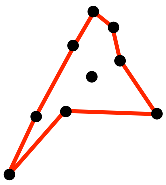
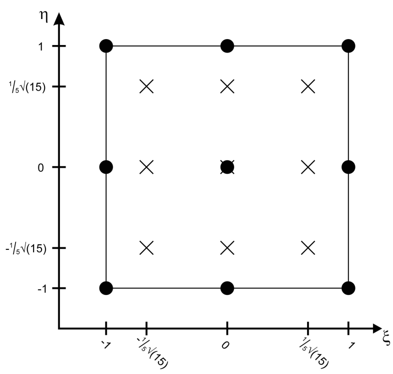
Distorted vs. idealized element

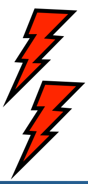
$$\int_0^{dx} \begin{bmatrix} \frac{\partial N_i(x)}{\partial x}\frac{\partial N_i(x)}{\partial x} & \frac{\partial N_i(x)}{\partial x}\frac{\partial N_{i+1}(x)}{\partial x} \\ \frac{\partial N_{i+1}(x)}{\partial x}\frac{\partial N_i(x)}{\partial x} & \frac{\partial N_{i+1}(x)}{\partial x}\frac{\partial N_{i+1}(x)}{\partial x} \end{bmatrix} A dx \begin{Bmatrix} u_i \\ u_{i+1} \end{Bmatrix} - \int_0^{dx} \begin{Bmatrix} N_i(x) \\ N_{i+1}(x) \end{Bmatrix} B dx = 0$$

1D: FEM introduction





- Derivatives of shape functions with respect to global coordinates
- Integral form written in terms of global coordinates (dx)



- Shape functions given in terms of local coordinates ξ
- Numerical integration more convenient in a local coordinate system.

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Two transformations are necessary

- Transform locally defined derivatives of shape functions to global coordinate system
- Transform locally performed (numerical) integration to global coordinates

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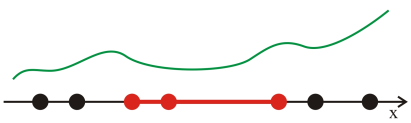
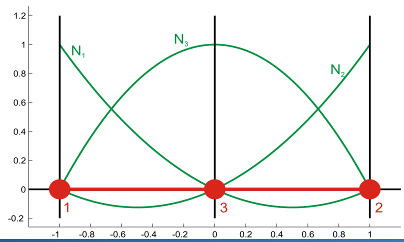
First transformation in 1D

$\longrightarrow$ Derivatives of shape fcts. from local to global

- Global distorted element
  - Coordinate x arbitrary
  - Derivatives of shape functions wanted here



- Local isoparametric element
  - Coordinate ξ from -1 to 1
  - Shape functions defined here
  - Derivatives of shape functions determinable here



First transformation in 1D

⟶ Derivatives of shape fcts. from local to global

- Global distorted element
  - Coordinate x arbitrary
  - Derivatives of shape functions wanted here

- Local isoparametric element
  - Coordinate ξ from -1 to 1
  - Shape functions defined here
  - Derivatives of shape functions determinable here

$$\mathbf{N}(\xi)=\begin{Bmatrix} N_1(\xi) \\ N_2(\xi) \\ N_3(\xi) \end{Bmatrix} \qquad \frac{\partial \mathbf{N}(\xi)}{\partial \xi}=\begin{Bmatrix} \partial N_1(\xi)/\partial \xi \\ \partial N_2(\xi)/\partial \xi \\ \partial N_3(\xi)/\partial \xi \end{Bmatrix}$$

First transformation in 1D

⟶ Derivatives of shape fcts. from local to global

Isoparametric element

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```
for intp=1:no_intpoi % ============================ INT.PTS. LOOP ======================================
    wtx          = Weight(intp);                                              % weight
    % Get parameters to perform coordinate transformation from natural to global element coordinates
        DHDS         = DHDS_A(:,:,intp)';
        jacob        = DHDS*COORD;                                           % compute Jacobian
        detjacob     = det(jacob);                                           % determinant of Jacobian
        invjacob     = inv(jacob);                                           % inverse of Jacobian matrix
        DHDX         = invjacob*DHDS;                                        % derivatives w.r.t. real coordinates

    % Compute kinematic/strain matrix
        ii           =([1:no_node_perel]-1)*no_dof+1;                        % working indexes arrays
        B(1,ii  )    = DHDX(1,:);
        B(2,ii+1)    = DHDX(2,:);
        B(3,ii  )    = DHDX(2,:);
        B(3,ii+1)    = DHDX(1,:);

    % STRAIN RATES
        STRAIN_RATES(:,intp,iel)                = B*A_old(Index);

    E                = MATPROP(1,Phase(iel));
    nu               = MATPROP(2,Phase(iel));
    prefac           = E/((1+nu)*(1-2*nu));
    D                = prefac * [ 1-nu nu 0; nu 1-nu 0; 0 0 (1-2*nu)/2];

    % STRESS UPDATE
        STRESSES(:,intp,iel)          = STRESSES_OLD(:,intp,iel) + D*B*A_old(Index);

    % CORRDINATES
        STRESS_GCOORD(1,intp,iel)     = H(:,intp)'*COORD_OLD(:,1);            % X-coordinate
        STRESS_GCOORD(2,intp,iel)     = H(:,intp)'*COORD_OLD(:,2);            % Y-coordinate

    % LOCAL STIFFNESS MATRIX
        K                             = K +( B'*D*B )*wtx*detjacob;           % element stiffness matrix

    % RHS VECTOR
        F_v(Index_v_local)            = F_v(Index_v_local) - ( B'* (STRESSES_OLD(:,intp,iel)) )*wtx*detjacob;
        F_v(Index_v_local(ii+1))      = F_v(Index_v_local(ii+1))-( MATPROP(3,Phase(iel))*gravity *H(:,intp))*wtx*detjacob;
end % ============================================= END OF INT.PTS. LOOP ====================================
```


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Second transformation in 1D

⟶ Integration form from local to global

- Global distorted element
  - Coordinate x arbitrary
  - Integral form of system of equations given here

- Local isoparametric element
  - Coordinate ξ from -1 to 1
  - Numerical integration performed here

```
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        DHDX        = invjacob*DHDS;                                        % derivatives w.r.t. real coordinates

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        F_v(Index_v_local(ii+1))      = F_v(Index_v_local(ii+1))-( MATPROP(3,Phase(iel))*gravity *H(:,intp))*wtx*detjacob;
end % ============================================= END OF INT.PTS. LOOP ====================================
```
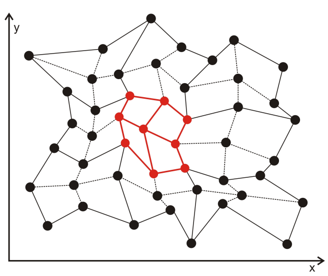


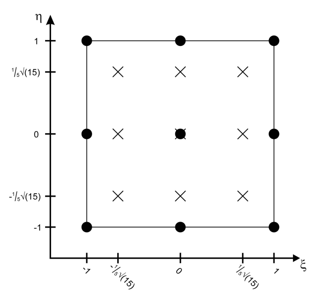
First transformation in 2D

⟶ Derivatives of shape fcts. from local to global

- Global distorted element
  - Coordinate x and y arbitrary
  - Derivatives of shape functions wanted here



- Local isoparametric element
  - Coordinate ξ and η from -1 to 1
  - Shape functions and their derivatives defined here



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First transformation in 2D

$\longrightarrow$ Derivatives of shape fcts. from local to global

- Global distorted element
  - Coordinate x and y arbitrary
  - Derivatives of shape functions wanted here

- Local isoparametric element
  - Coordinate ξ and η from -1 to 1
  - Shape functions and their derivatives defined here

$$\mathbf{N}(\xi,\eta)=\begin{Bmatrix} N_1(\xi,\eta) \\ N_2(\xi,\eta) \\ \dots \\ N_9(\xi,\eta) \end{Bmatrix}$$

$$\nabla_{\xi,\eta}\mathbf{N}(\xi,\eta)=\begin{bmatrix} \frac{\partial N_1(\xi,\eta)}{\partial\xi} & \frac{\partial N_1(\xi,\eta)}{\partial\eta} \\ \frac{\partial N_2(\xi,\eta)}{\partial\xi} & \frac{\partial N_2(\xi,\eta)}{\partial\eta} \\ \dots & \dots \\ \frac{\partial N_9(\xi,\eta)}{\partial\xi} & \frac{\partial N_9(\xi,\eta)}{\partial\eta} \end{bmatrix}$$

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First transformation in 2D

$\longrightarrow$ Derivatives of shape fcts. from local to global

- Derivation of the Jacobian in a FEM manner!
  - Definition $\mathbf{J}=\begin{bmatrix} \partial x/\partial\xi & \partial y/\partial\xi \\ \partial y/\partial\eta & \partial x/\partial\eta \end{bmatrix}$
  - So

$$\mathbf{J}=\begin{bmatrix} \partial x/\partial\xi & \partial y/\partial\xi \\ \partial y/\partial\eta & \partial x/\partial\eta \end{bmatrix}=\nabla_{\xi,\eta}\mathbf{N}^T(\xi,\eta)\mathbf{x}=\begin{bmatrix} \frac{\partial N_1(\xi,\eta)}{\partial\xi} & \frac{\partial N_2(\xi,\eta)}{\partial\xi} & \dots & \frac{\partial N_9(\xi,\eta)}{\partial\xi} \\ \frac{\partial N_1(\xi,\eta)}{\partial\eta} & \frac{\partial N_2(\xi,\eta)}{\partial\eta} & \dots & \frac{\partial N_9(\xi,\eta)}{\partial\eta} \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ \dots \\ x_9 \end{Bmatrix}$$

Second transformation in 2D

$\longrightarrow$ Integration form from local to global

- Global distorted element
  - Coordinate x and y arbitrary
  - Integral form of system of equations given here
- Local isoparametric element
  - Coordinate ξ and η from -1 to 1
  - Numerical integration performed here