Heat equation:

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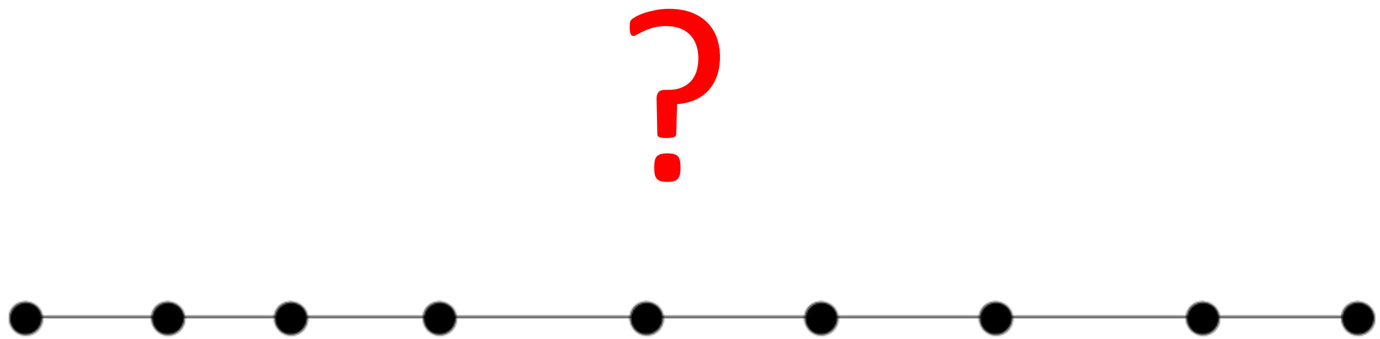
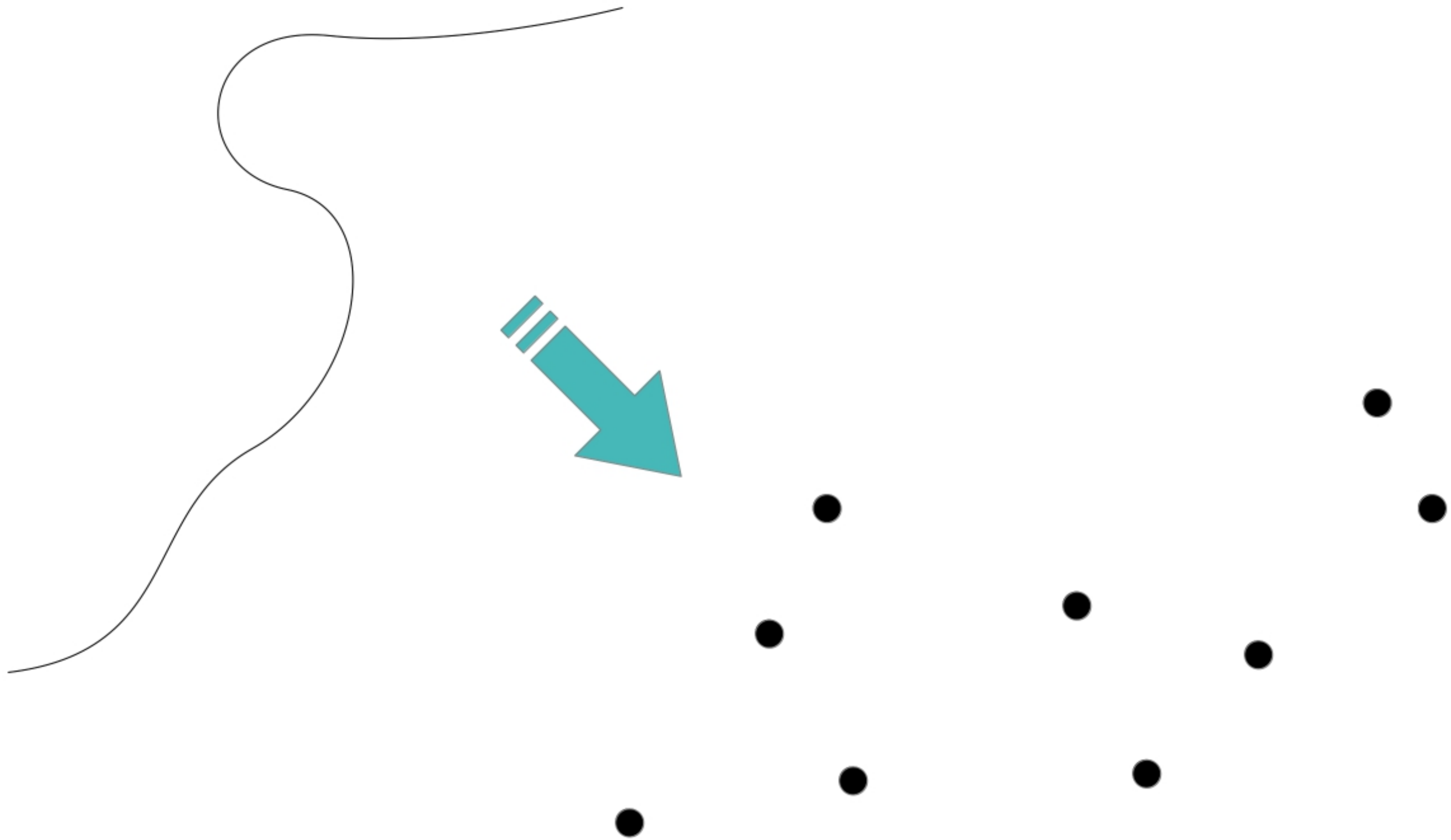
Numerical solution
requires a discretisation
In both space and time





Example: Fourier series

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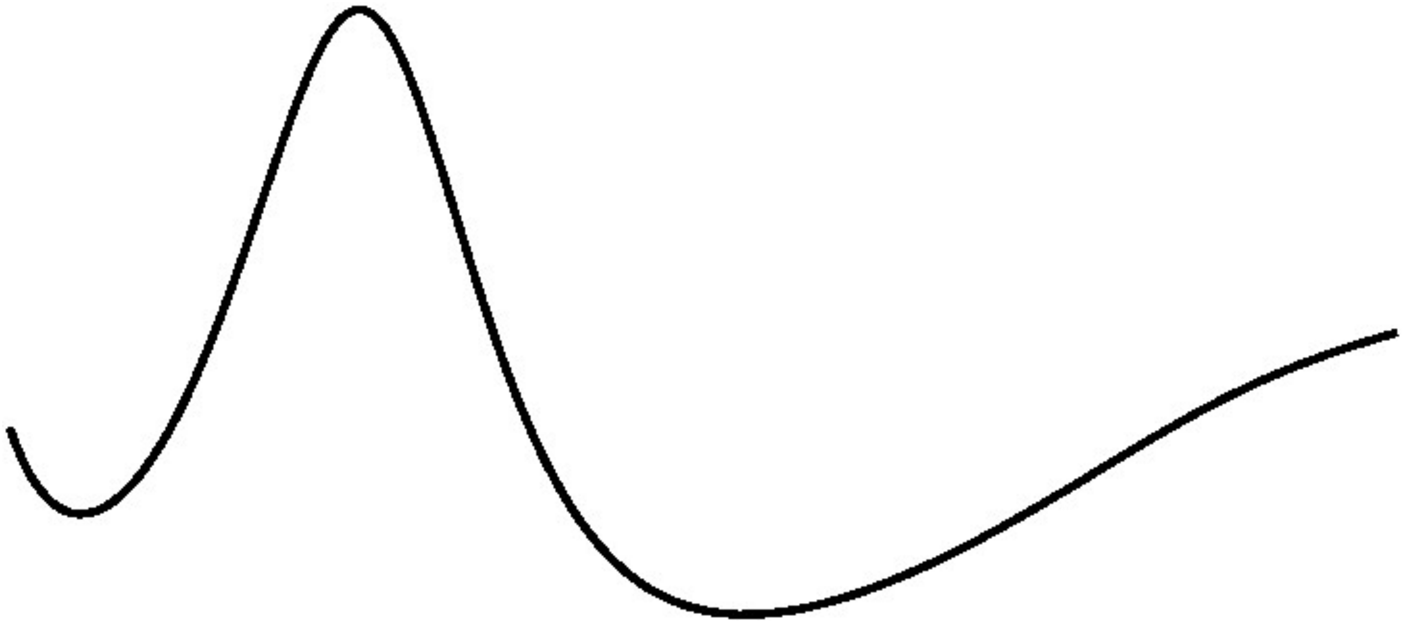
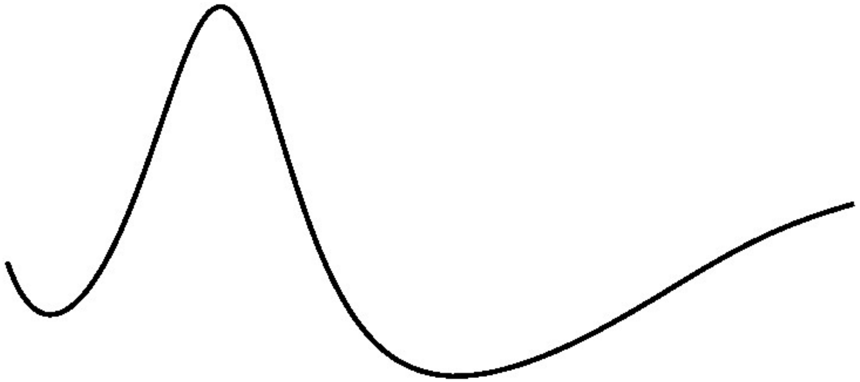
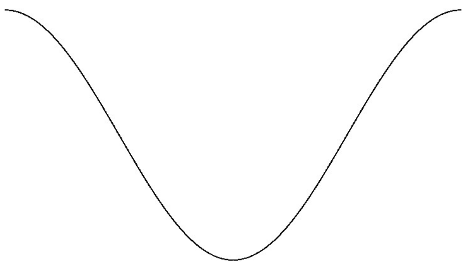
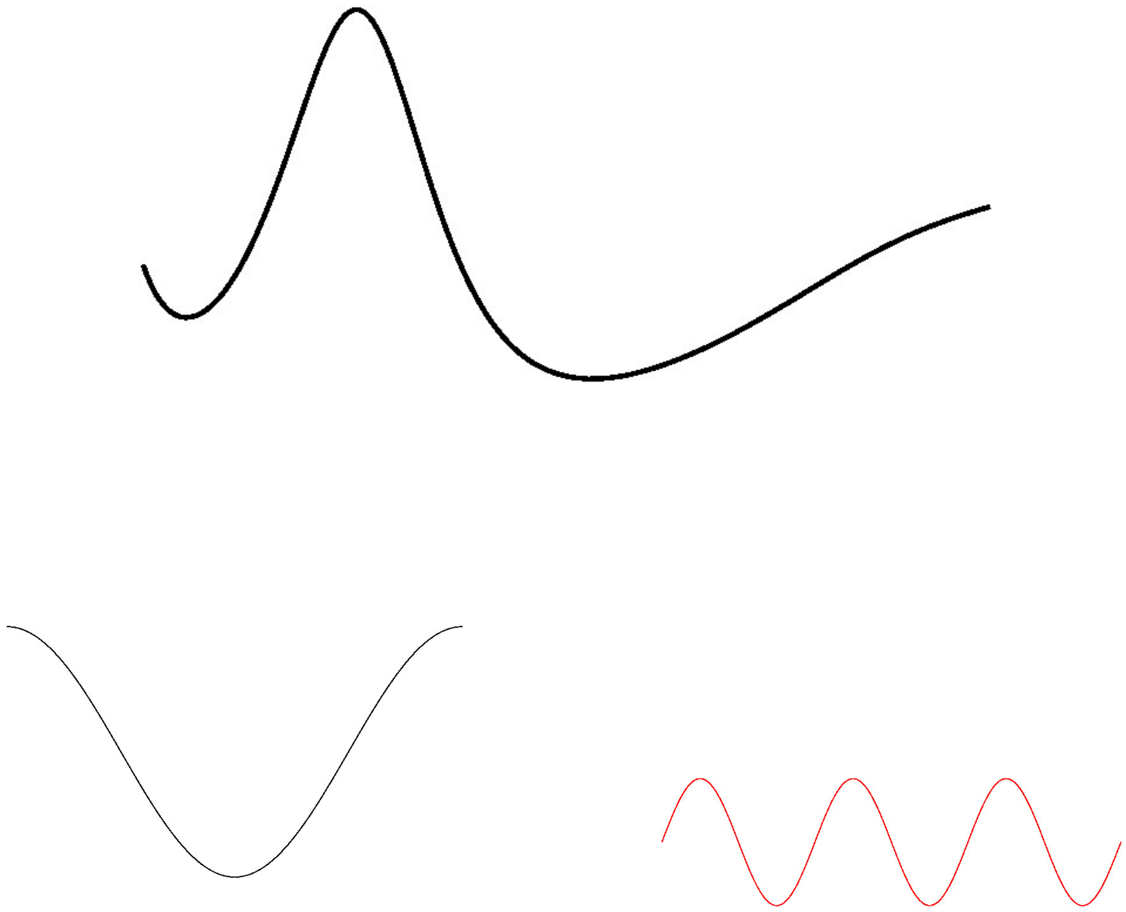


Example: Fourier series

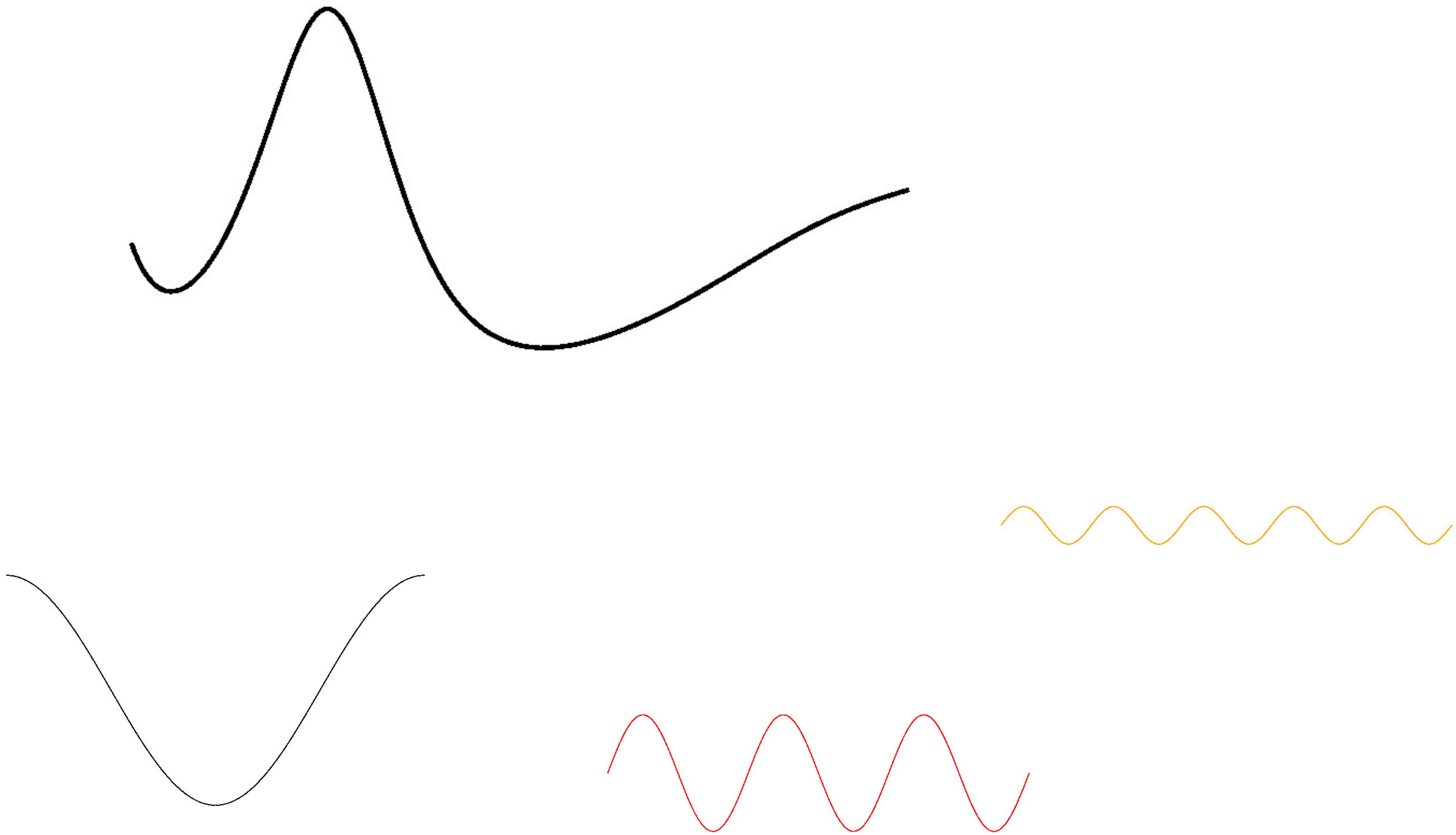




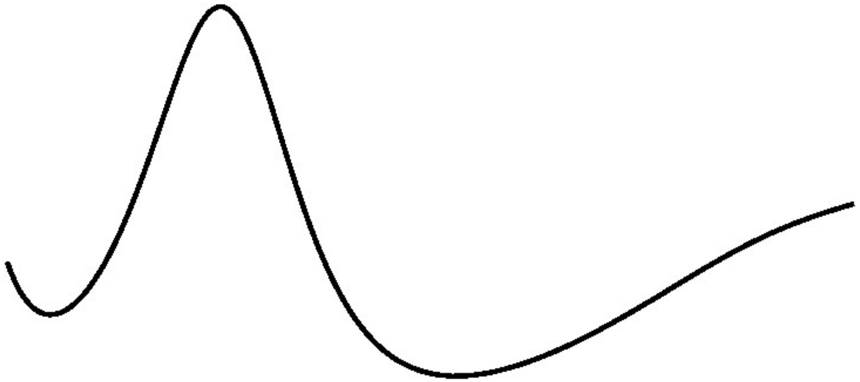
Example: Fourier series

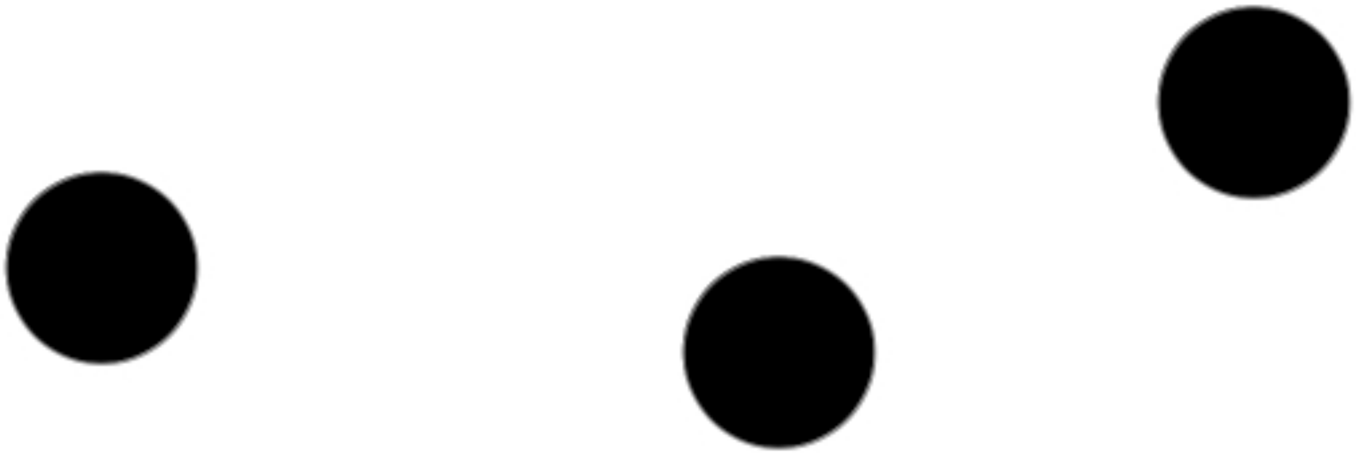


Example: Fourier series



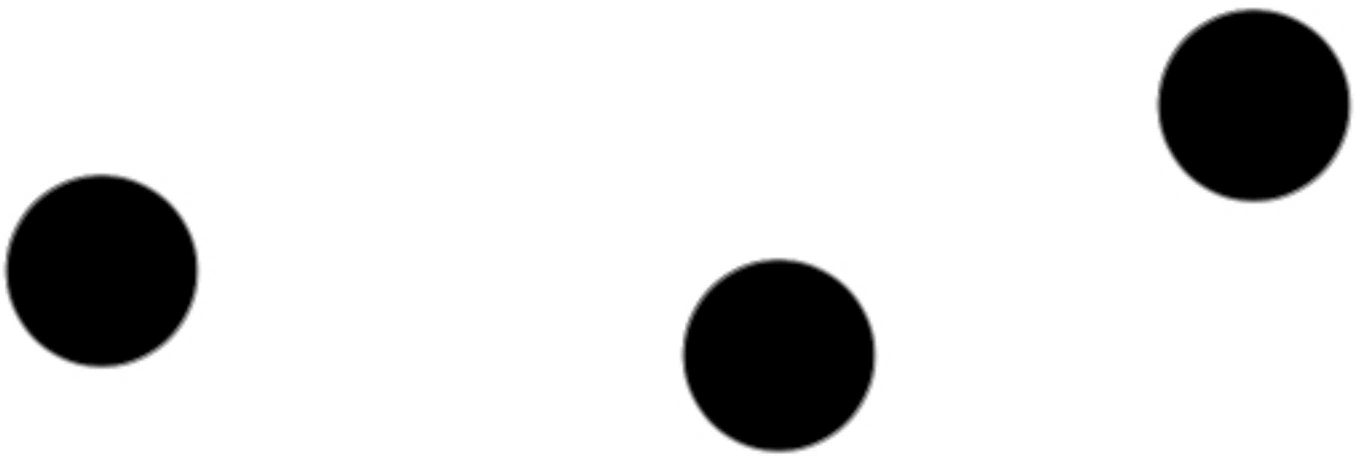
Example: Fourier series





Example: Fourier series

$$s_N(x) = \overbrace{a_0}^{A_0}/2 + \sum_{n=1}^{N} \left(\overbrace{a_n}^{A_n\ \sin(\phi_n)} \cos\left(\frac{2\pi n x}{P}\right) + \overbrace{b_n}^{A_n\ \cos(\phi_n)} \sin\left(\frac{2\pi n x}{P}\right) \right)$$



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$$a_n = \frac{2}{P} \int_{x_0}^{x_0+P} s(x) \cdot \cos\left(\frac{2\pi n x}{P}\right) dx$$

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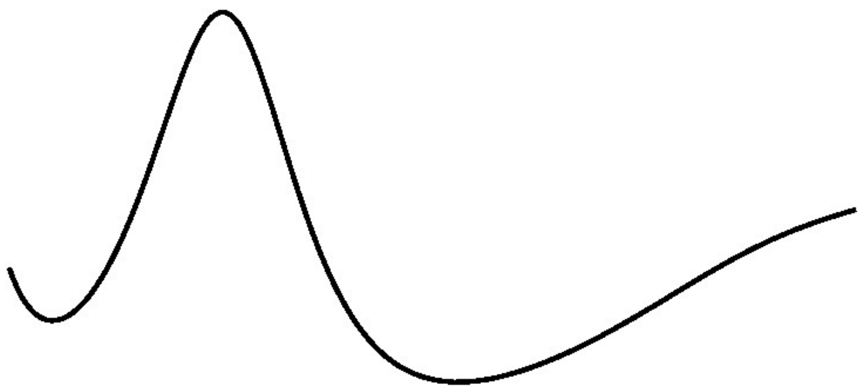
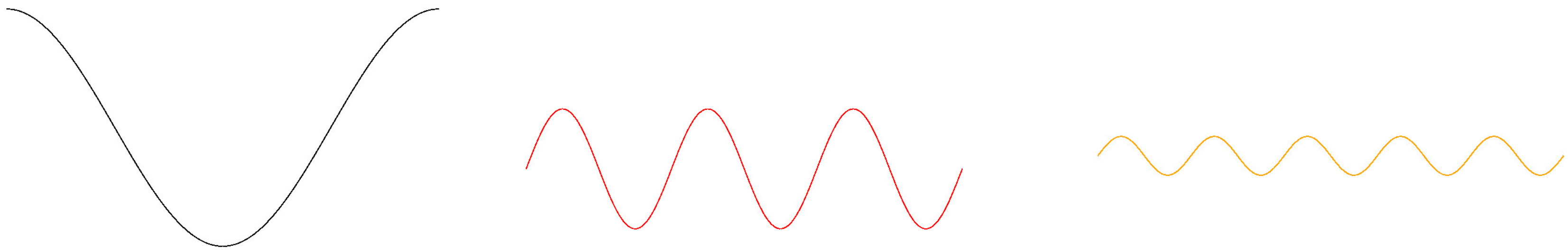
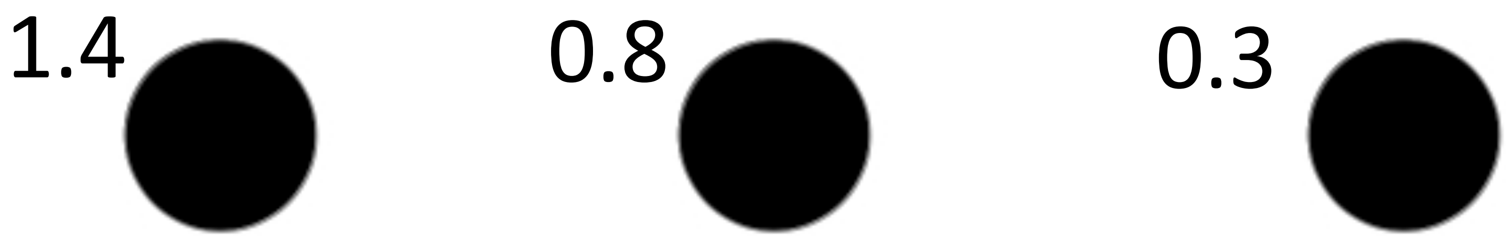
Arbitrary function



Orthogonal basis functions



Spectral coefficients



Finite elements?

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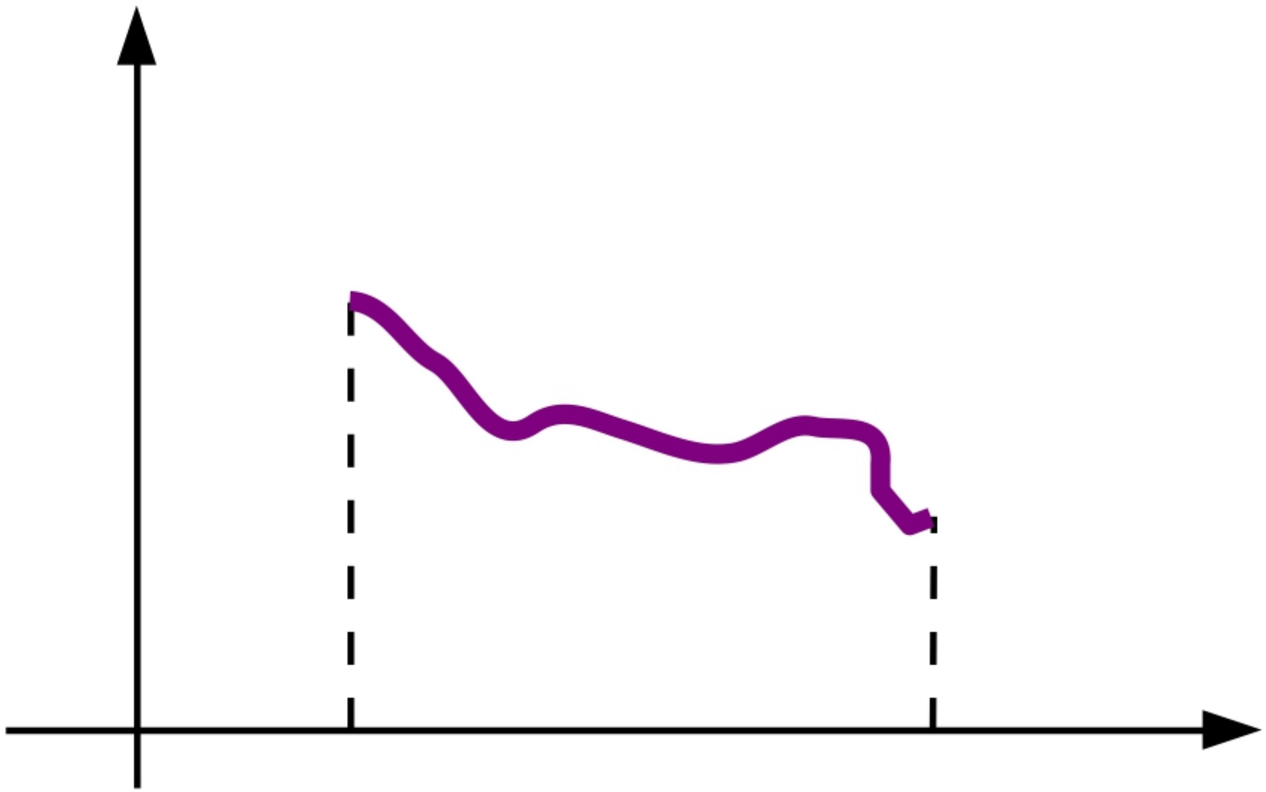
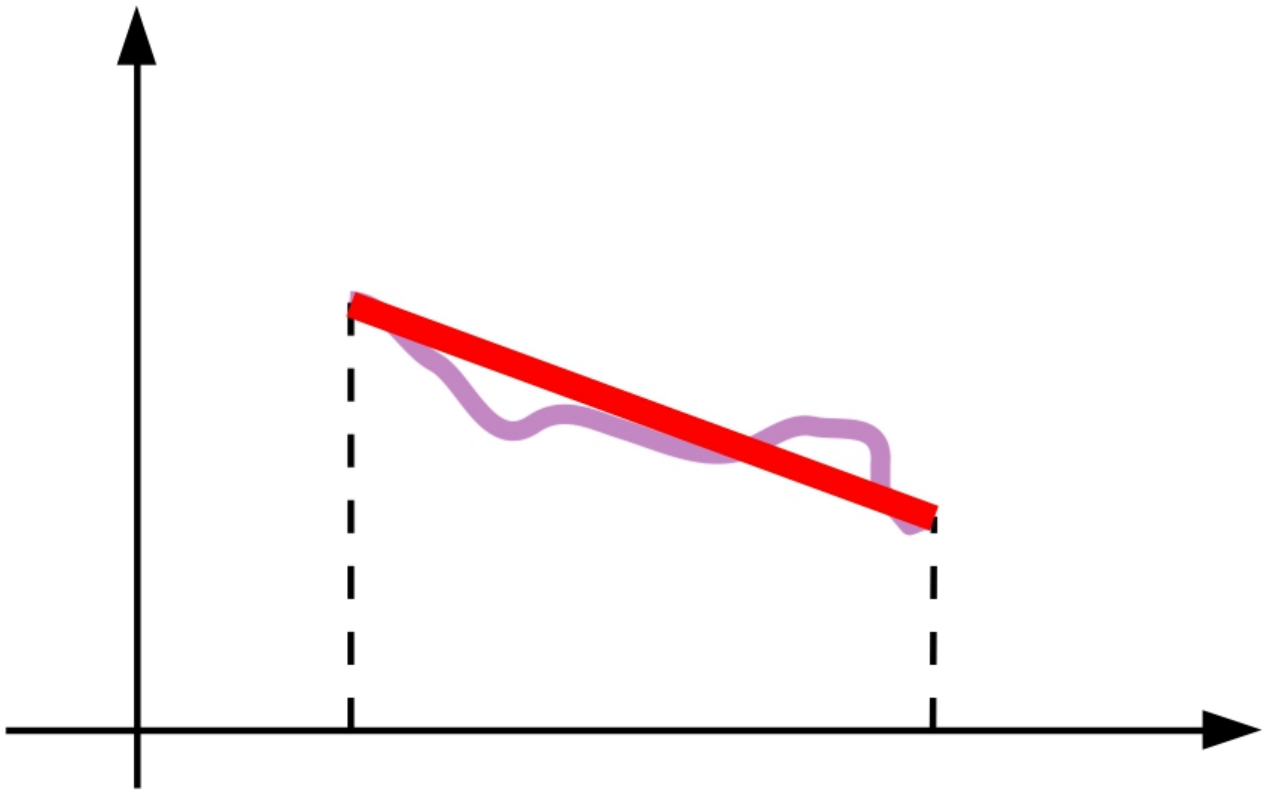
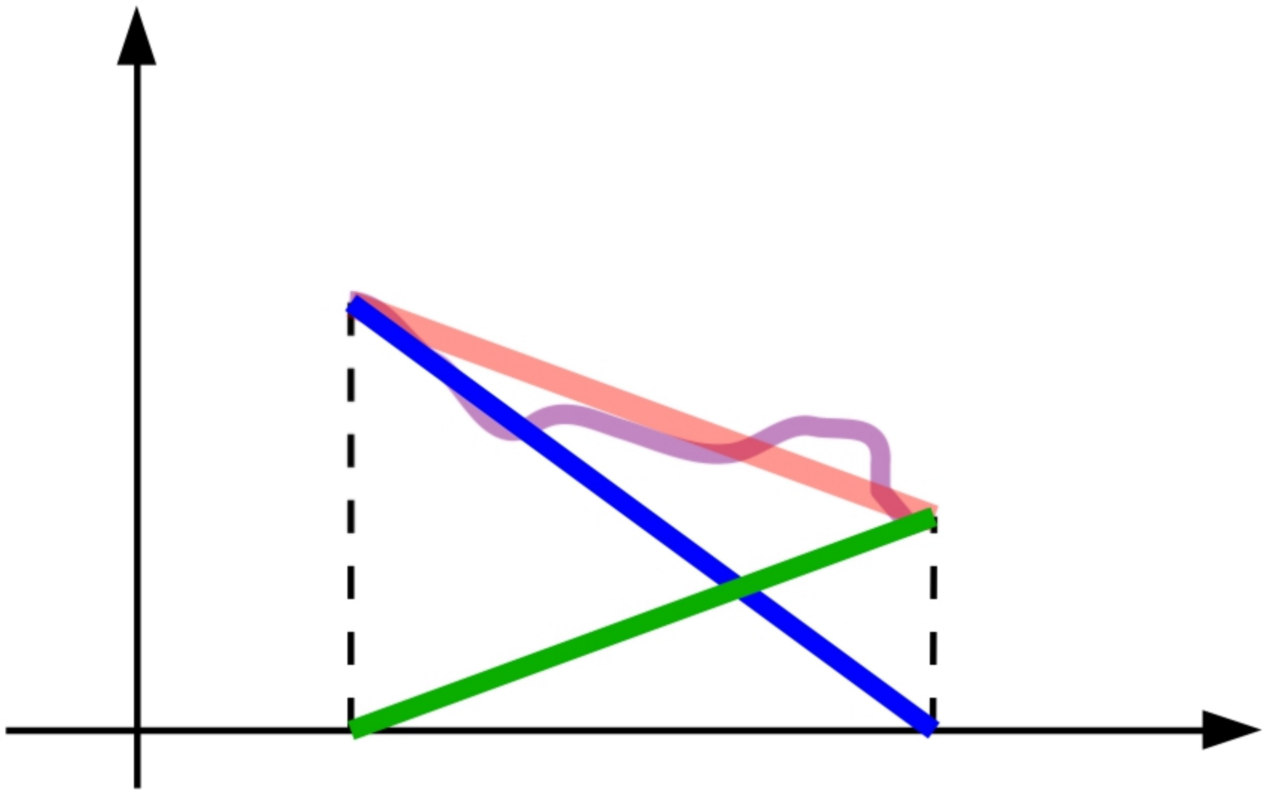


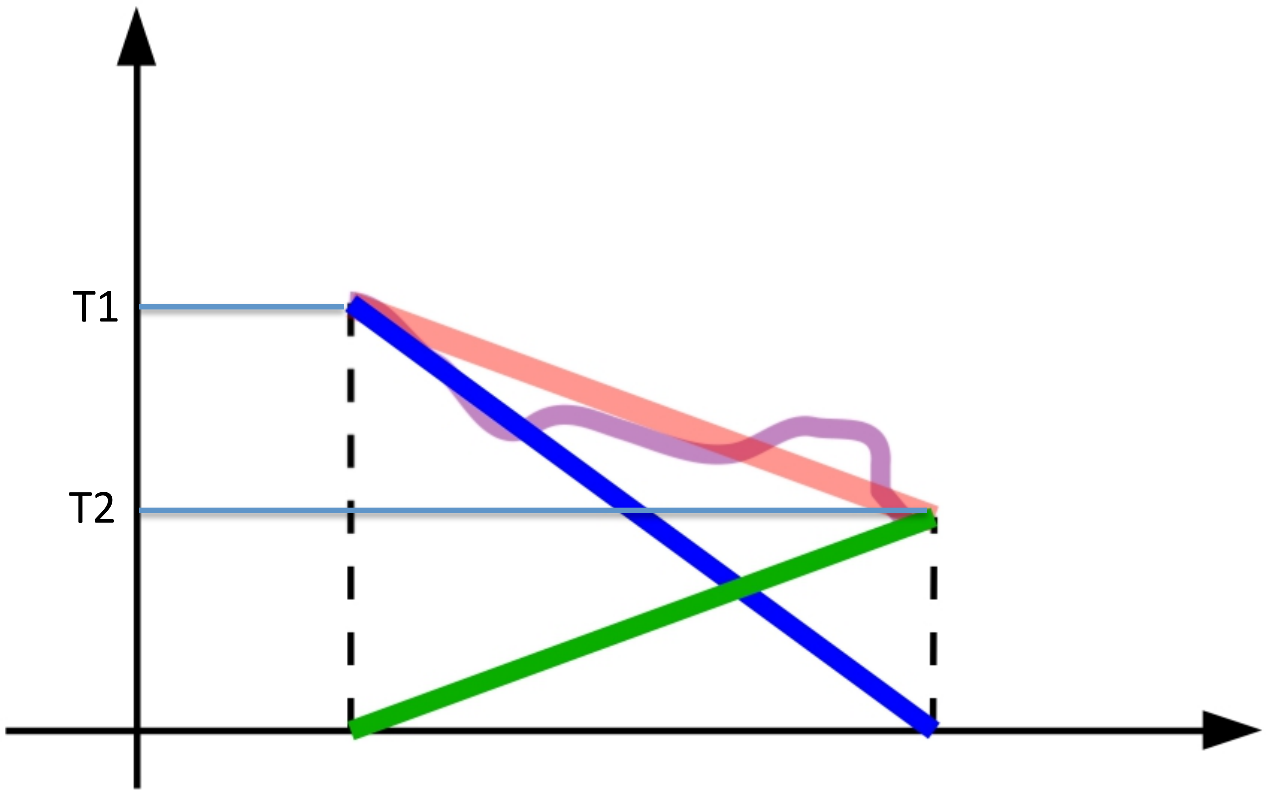
Finite elements?



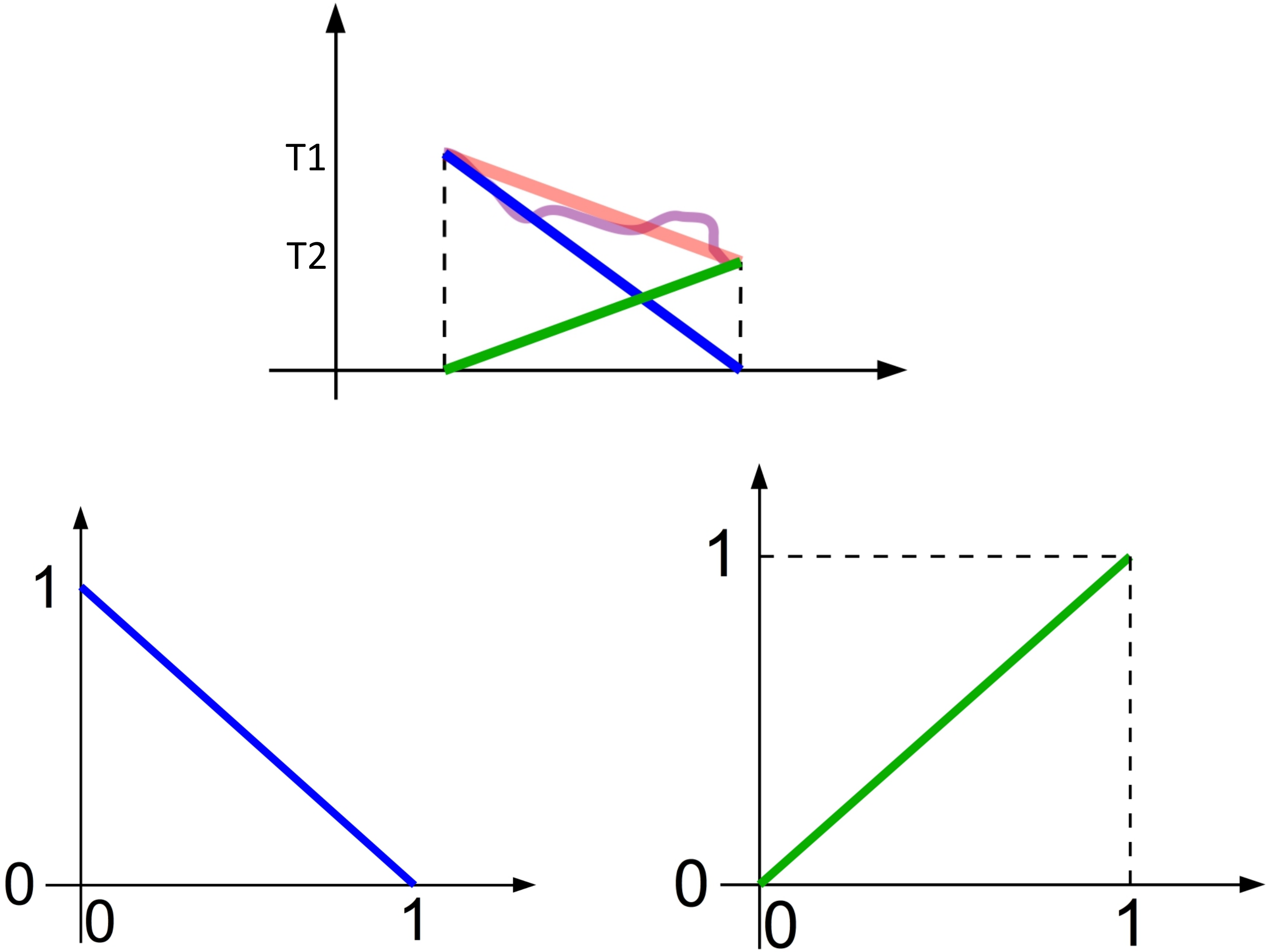
Finite elements?



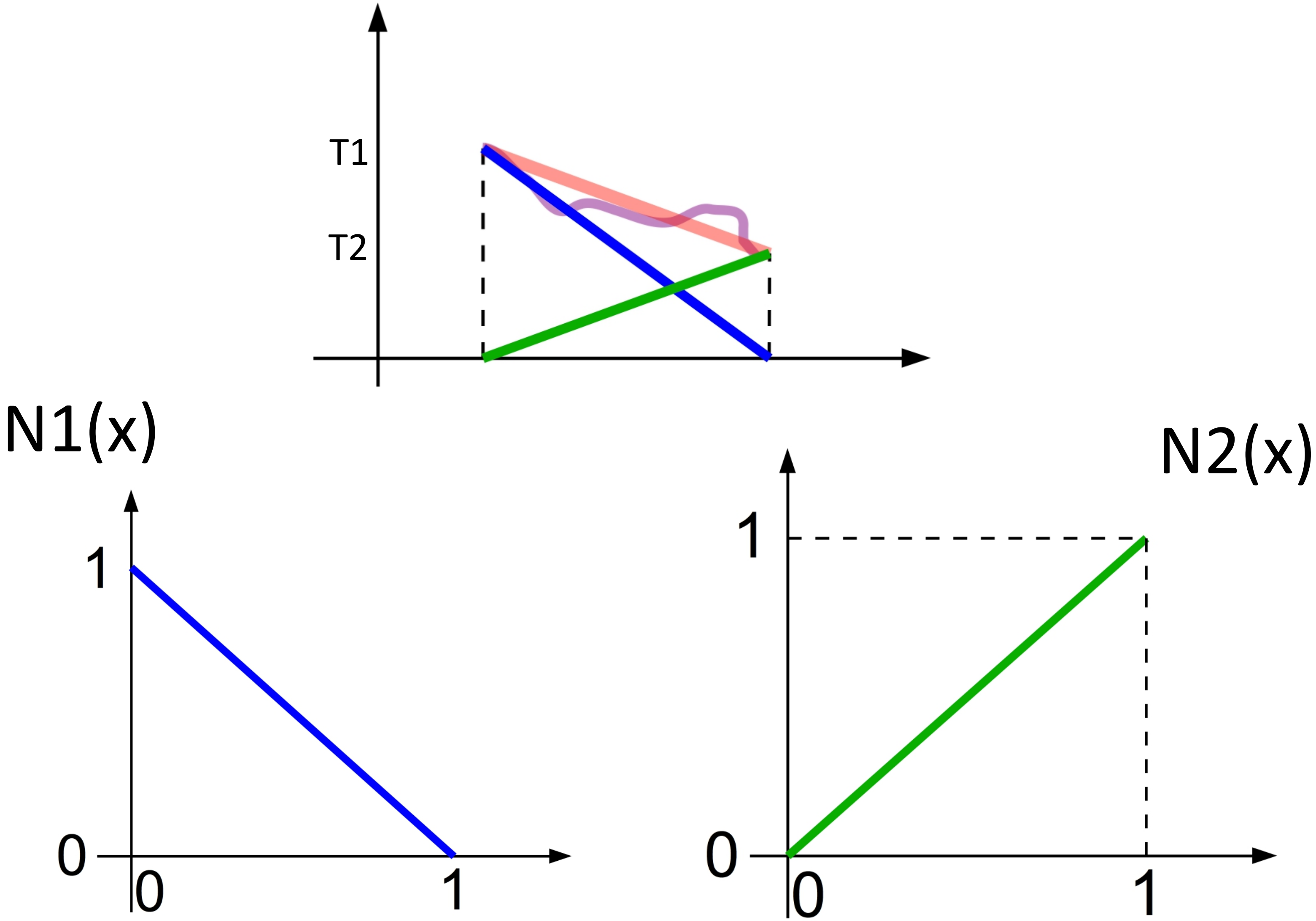
Finite elements?



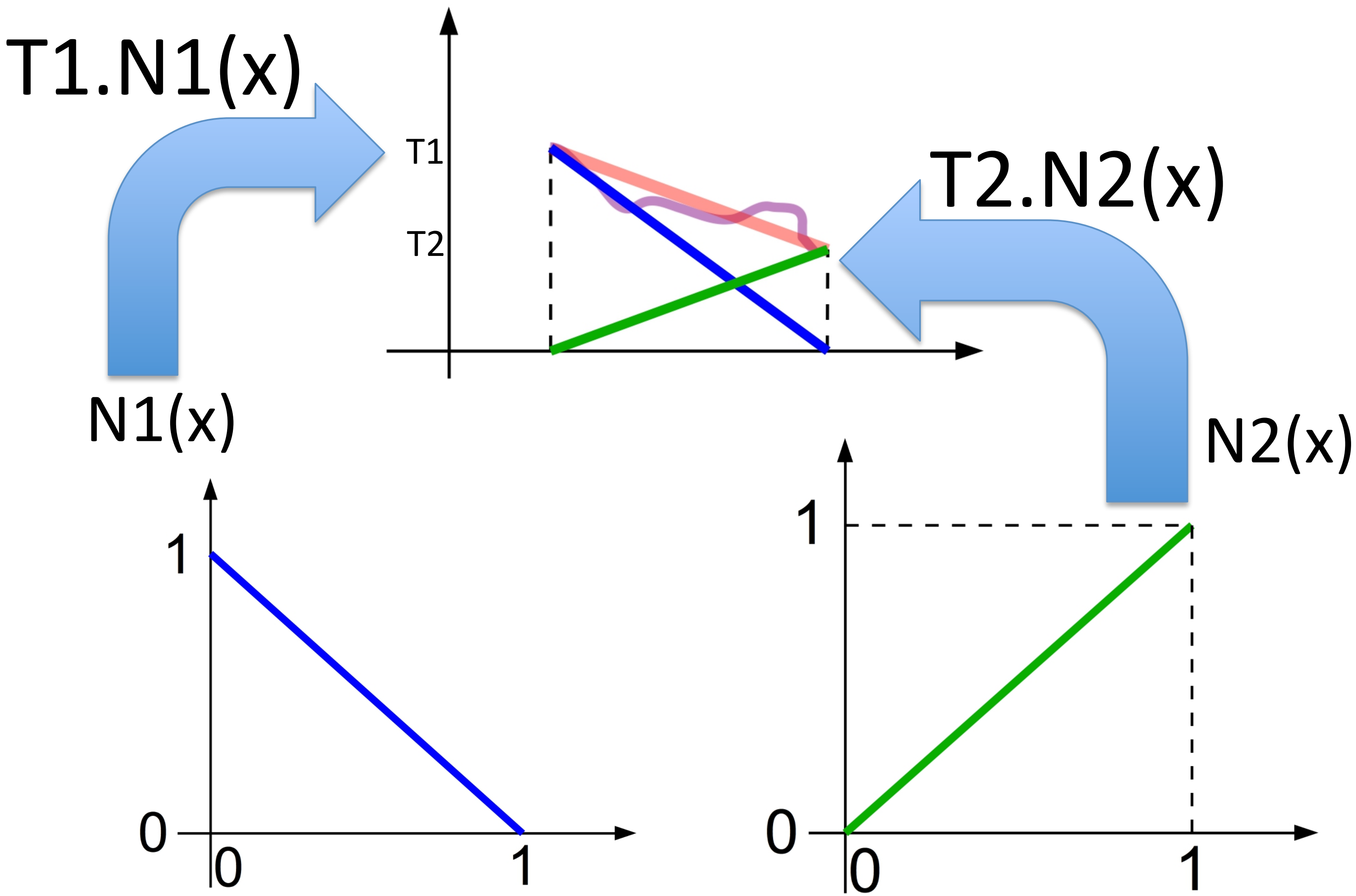
Finite elements?



Finite elements?

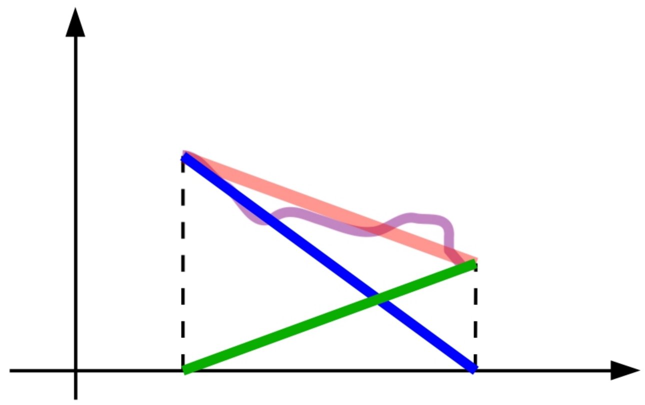


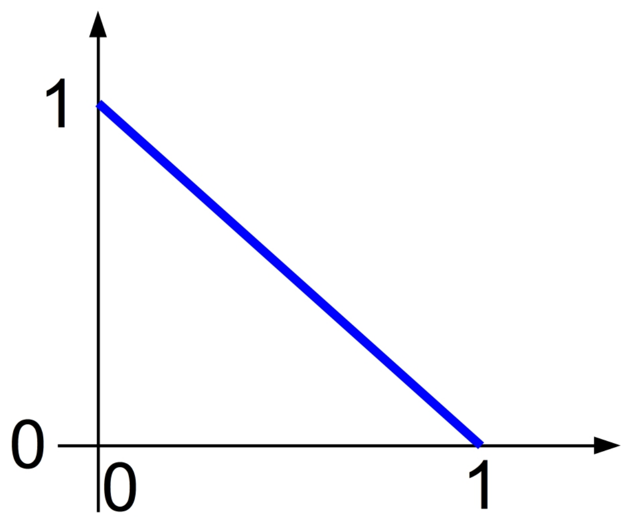
Finite elements?

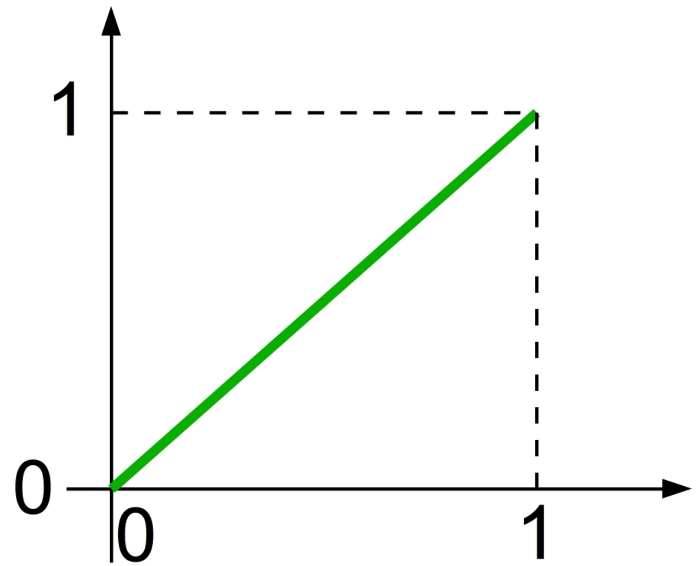


$$T(x) \approx N_1(x)T_1 + N_2(x)T_2$$

$$T(x) \approx \begin{bmatrix} N_1(x) & N_2(x) \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \mathbf{NT}$$







$$N_1(x) = 1 - \frac{x}{L}, \qquad N_2(x) = \frac{x}{L},$$

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\kappa(x)\frac{\partial T}{\partial x}\right) + s(x)$$

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$$\frac{\partial}{\partial t}\left(\begin{bmatrix} N_1(x) & N_2(x) \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}\right) - \frac{\partial}{\partial x}\left(\kappa(x)\frac{\partial}{\partial x}\left(\begin{bmatrix} N_1(x) & N_2(x) \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}\right)\right) - s(x) = R$$

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Insufficient equation

-> R is unknown!

We look for the solution such that the projection of the residual over the basis functions is zero

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$$\frac{\partial}{\partial t}\left(\begin{bmatrix} N_1(x) & N_2(x) \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \end{bmatrix}\right) - \frac{\partial}{\partial x}\left(\kappa(x)\frac{\partial}{\partial x}\left(\begin{bmatrix} N_1(x) & N_2(x) \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \end{bmatrix}\right)\right) - s(x) = R$$

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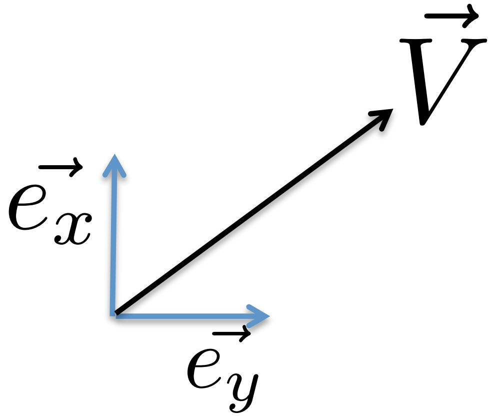
$$\frac{\partial}{\partial t}\left(\begin{bmatrix} N_1(x) & N_2(x) \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \end{bmatrix}\right) - \frac{\partial}{\partial x}\left(\kappa(x)\frac{\partial}{\partial x}\left(\begin{bmatrix} N_1(x) & N_2(x) \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \end{bmatrix}\right)\right) - s(x) = R$$

$$\int_0^L \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \frac{\partial}{\partial t}\left(\begin{bmatrix} N_1 & N_2 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \end{bmatrix}\right) dx - \int_0^L \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \frac{\partial}{\partial x}\left(\kappa(x)\frac{\partial}{\partial x}\left(\begin{bmatrix} N_1 & N_2 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \end{bmatrix}\right)\right) dx - \int_0^L \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} s(x)\, dx = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Projections

Common space:



$$\vec{V} = V_x \vec{e_x} + V_y \vec{e_y}$$

$$V_x = \vec{V} \cdot \vec{e_x}$$

$$V_y = \vec{V} \cdot \vec{e_y}$$

Fourier:

$$a_n = \frac{2}{P} \int_{x_0}^{x_0+P} s(x) \cdot \cos\left(\frac{2\pi n x}{P}\right) dx$$

$$b_n = \frac{2}{P} \int_{x_0}^{x_0+P} s(x) \cdot \sin\left(\frac{2\pi n x}{P}\right) dx$$

FEM:

$$\int_0^L \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} s(x)\, dx$$

$$
\int_0^L \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \frac{\partial}{\partial t} \left(\begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \right) dx
$$

$$
- \int_0^L \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \frac{\partial}{\partial x} \left(\kappa(x) \frac{\partial}{\partial x} \left(\begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \right) \right) dx - \int_0^L \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} s(x)\, dx = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

$$\int_0^L \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \frac{\partial}{\partial t} \left(\begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \right) dx$$

$$- \int_0^L \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \frac{\partial}{\partial x} \left(\kappa(x) \frac{\partial}{\partial x} \left(\begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \right) \right) dx - \int_0^L \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} s(x)\, dx = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\int_\Omega N_i \frac{\partial}{\partial x} \left(\kappa(x) \frac{\partial N_j}{\partial x} \right) dx = - \int_\Omega \kappa(x) \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x}\, dx + \kappa(x) N_i \frac{\partial N_j}{\partial x} \bigg|_{x_A}^{x_B}$$

$$\int_0^L \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \frac{\partial}{\partial t} \left(\begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \right) dx$$

$$- \int_0^L \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \frac{\partial}{\partial x} \left(\kappa(x) \frac{\partial}{\partial x} \left(\begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \right) \right) dx - \int_0^L \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} s(x)\, dx = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\int_\Omega N_i \frac{\partial}{\partial x} \left(\kappa(x) \frac{\partial N_j}{\partial x} \right) dx = - \int_\Omega \kappa(x) \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x}\, dx + \kappa(x) N_i \frac{\partial N_j}{\partial x} \Bigg|_{x_A}^{x_B}$$

$$\begin{bmatrix} \int_0^L N_1 N_1\, dx & \int_0^L N_1 N_2\, dx \\ \int_0^L N_2 N_1\, dx & \int_0^L N_2 N_2\, dx \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$$

$$+ \bar{\kappa} \begin{bmatrix} \int_0^L \frac{\partial N_1}{\partial x} \frac{\partial N_1}{\partial x}\, dx & \int_0^L \frac{\partial N_1}{\partial x} \frac{\partial N_2}{\partial x}\, dx \\ \int_0^L \frac{\partial N_2}{\partial x} \frac{\partial N_1}{\partial x}\, dx & \int_0^L \frac{\partial N_2}{\partial x} \frac{\partial N_2}{\partial x}\, dx \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$$

$$- \bar{s} \begin{bmatrix} \int_0^L N_1\, dx \\ \int_0^L N_2\, dx \end{bmatrix} - \begin{bmatrix} N_1 q_A \Big|_{x_A} \\ N_2 q_B \Big|_{x_B} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{L}{3} & \frac{L}{6} \\ \frac{L}{6} & \frac{L}{3} \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} + \bar{\kappa} \begin{bmatrix} \frac{1}{L} & -\frac{1}{L} \\ -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} - \bar{s} \begin{bmatrix} \frac{L}{2} \\ \frac{L}{2} \end{bmatrix} - \begin{bmatrix} N_1 q_A \big|_{x_A} \\ N_2 q_B \big|_{x_B} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{L}{3} & \frac{L}{6} \\ \frac{L}{6} & \frac{L}{3} \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} + \bar{\kappa} \begin{bmatrix} \frac{1}{L} & -\frac{1}{L} \\ -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} - \bar{s} \begin{bmatrix} \frac{L}{2} \\ \frac{L}{2} \end{bmatrix} - \begin{bmatrix} N_1 q_A \big|_{x_A} \\ N_2 q_B \big|_{x_B} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{M} \left(\frac{\partial \mathbf{T}}{\partial t} \right) + \mathbf{K}\,\mathbf{T} = \mathbf{F}$$

$$\begin{bmatrix} \frac{L}{3} & \frac{L}{6} \\ \frac{L}{6} & \frac{L}{3} \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} + \bar{\kappa} \begin{bmatrix} \frac{1}{L} & -\frac{1}{L} \\ -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} - \bar{s} \begin{bmatrix} \frac{L}{2} \\ \frac{L}{2} \end{bmatrix} - \begin{bmatrix} N_1 q_A \big|_{x_A} \\ N_2 q_B \big|_{x_B} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{M} \left(\frac{\partial \mathbf{T}}{\partial t} \right) + \mathbf{K}\,\mathbf{T} = \mathbf{F}$$

$$\mathbf{M} = \begin{bmatrix} \dfrac{L}{3} & \dfrac{L}{6} \\ \dfrac{L}{6} & \dfrac{L}{3} \end{bmatrix} \qquad \mathbf{K} = \bar{\kappa} \begin{bmatrix} \dfrac{1}{L} & -\dfrac{1}{L} \\ -\dfrac{1}{L} & \dfrac{1}{L} \end{bmatrix} \qquad \mathbf{F} = \bar{s} \begin{bmatrix} \dfrac{L}{2} \\ \dfrac{L}{2} \end{bmatrix} + \begin{bmatrix} N_1 q_A \Big|_{x_A} \\ N_2 q_B \Big|_{x_B} \end{bmatrix}$$

Solving for the time-dependence

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$$\mathbf{M}\left(\frac{\mathbf{T}^{n+1}-\mathbf{T}^{n}}{\Delta t}\right)+\mathbf{K}\,\mathbf{T}^{n+1}=\mathbf{F}$$

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$$\mathbf{M}\left(\frac{\mathbf{T}^{n+1}-\mathbf{T}^{n}}{\Delta t}\right)+\mathbf{K}\,\mathbf{T}^{n+1}=\mathbf{F}$$

$$\left(\frac{1}{\Delta t}\mathbf{M}+\mathbf{K}\right)\mathbf{T}^{n+1}=\frac{1}{\Delta t}\mathbf{M}\,\mathbf{T}^{n}+\mathbf{F}$$

Solving for the time-dependence

$$\mathbf{M}\left(\frac{\mathbf{T}^{n+1}-\mathbf{T}^{n}}{\Delta t}\right)+\mathbf{K}\,\mathbf{T}^{n+1}=\mathbf{F}$$

$$\left(\frac{1}{\Delta t}\mathbf{M}+\mathbf{K}\right)\mathbf{T}^{n+1}=\frac{1}{\Delta t}\mathbf{M}\,\mathbf{T}^{n}+\mathbf{F}$$

$$\mathbf{L}\,\mathbf{T}^{n+1}=\mathbf{R}\,\mathbf{T}^{n}+\mathbf{F}$$

Solving for the time-dependence

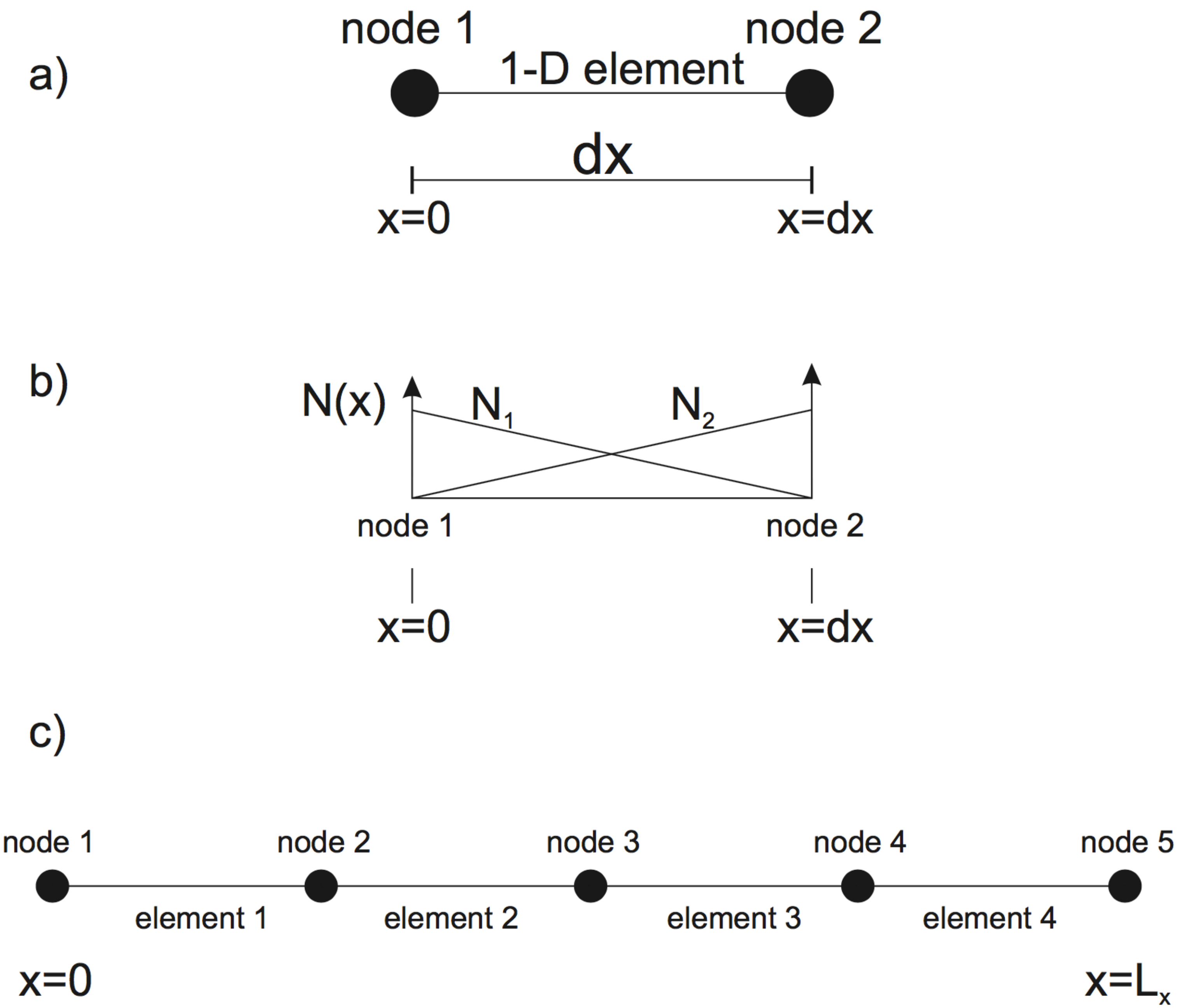
$$\mathbf{M}\left(\frac{\mathbf{T}^{n+1}-\mathbf{T}^{n}}{\Delta t}\right)+\mathbf{K}\,\mathbf{T}^{n+1}=\mathbf{F}$$

$$\left(\frac{1}{\Delta t}\mathbf{M}+\mathbf{K}\right)\mathbf{T}^{n+1}=\frac{1}{\Delta t}\mathbf{M}\,\mathbf{T}^{n}+\mathbf{F}$$

$$\mathbf{L}\,\mathbf{T}^{n+1}=\mathbf{R}\,\mathbf{T}^{n}+\mathbf{F}$$

$$\mathbf{L}=\begin{bmatrix}\frac{L}{3\Delta t}+\frac{\bar{\kappa}}{L} & \frac{L}{6\Delta t}-\frac{\bar{\kappa}}{L}\\ \frac{L}{6\Delta t}-\frac{\bar{\kappa}}{L} & \frac{L}{3\Delta t}+\frac{\bar{\kappa}}{L}\end{bmatrix} \qquad \mathbf{R}=\begin{bmatrix}\frac{L}{3\Delta t} & \frac{L}{6\Delta t}\\ \frac{L}{6\Delta t} & \frac{L}{3\Delta t}\end{bmatrix}$$

More than one element?


a)
node 1
node 2
1-D element
dx
x=0
x=dx
b)
N(x)
N1
N2
node 1
node 2
x=0
x=dx
c)
node 1
node 2
node 3
node 4
node 5
element 1
element 2
element 3
element 4
x=0
x=Lx

$$\mathbf{L}\,\mathbf{T}^{n+1} = \mathbf{R}\,\mathbf{T}^{n} + \mathbf{F}$$

$$\mathbf{L} = \begin{bmatrix} \frac{L}{3\Delta t} + \frac{\bar{\kappa}}{L} & \frac{L}{6\Delta t} - \frac{\bar{\kappa}}{L} \\ \frac{L}{6\Delta t} - \frac{\bar{\kappa}}{L} & \frac{L}{3\Delta t} + \frac{\bar{\kappa}}{L} \end{bmatrix} \qquad \mathbf{R} = \begin{bmatrix} \frac{L}{3\Delta t} & \frac{L}{6\Delta t} \\ \frac{L}{6\Delta t} & \frac{L}{3\Delta t} \end{bmatrix}$$

$$\mathbf{L} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \qquad \mathbf{R} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}$$

$$
\begin{bmatrix} L_{11} & L_{12} & 0 & 0 & 0 \\ L_{21} & L_{22}+L_{11} & L_{12} & 0 & 0 \\ 0 & L_{21} & L_{22}+L_{11} & L_{12} & 0 \\ 0 & 0 & L_{21} & L_{22}+L_{11} & L_{12} \\ 0 & 0 & 0 & L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix}^{n+1}
= \begin{bmatrix} R_{11} & R_{12} & 0 & 0 & 0 \\ R_{21} & R_{22}+R_{11} & R_{12} & 0 & 0 \\ 0 & R_{21} & R_{22}+R_{11} & R_{12} & 0 \\ 0 & 0 & R_{21} & R_{22}+R_{11} & R_{12} \\ 0 & 0 & 0 & R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix}^{n}
+ sL \begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \\ 1 \\ \frac{1}{2} \end{bmatrix} + \begin{bmatrix} q_A \\ 0 \\ 0 \\ 0 \\ q_B \end{bmatrix}
$$

Global matrices

$$\mathbf{L}_G\,\mathbf{T}^{n+1} = \mathbf{R}_G\,\mathbf{T}^n + \mathbf{F}_G$$

Global matrices

$$\mathbf{L}_G\,\mathbf{T}^{n+1} = \mathbf{R}_G\,\mathbf{T}^n + \mathbf{F}_G$$

$$\mathbf{L}_G\,\mathbf{T}^{n+1} = \mathbf{b}$$

Global matrices

$$\mathbf{L}_G\,\mathbf{T}^{n+1} = \mathbf{R}_G\,\mathbf{T}^n + \mathbf{F}_G$$

$$\mathbf{L}_G\,\mathbf{T}^{n+1} = \mathbf{b}$$

$$\mathbf{T}^{n+1} = \mathbf{L}_{\boldsymbol{G}}{}^{-1}\mathbf{b}$$