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$$\frac{\partial}{\partial t}\left(\begin{bmatrix} N_1(x) & N_2(x) \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}\right) - \frac{\partial}{\partial x}\left(\kappa(x)\frac{\partial}{\partial x}\left(\begin{bmatrix} N_1(x) & N_2(x) \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}\right)\right) - s(x) = R,$$

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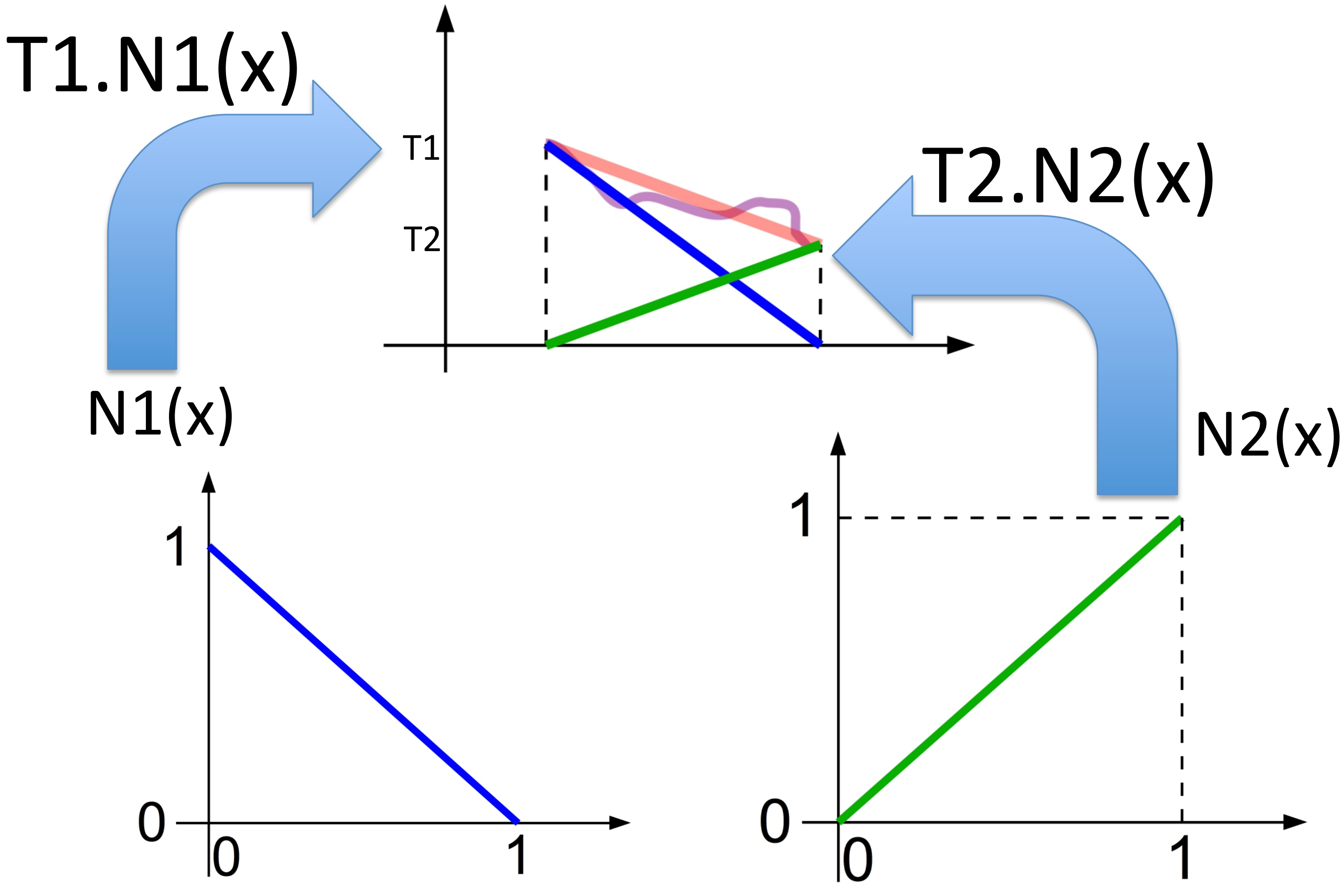
Weak form

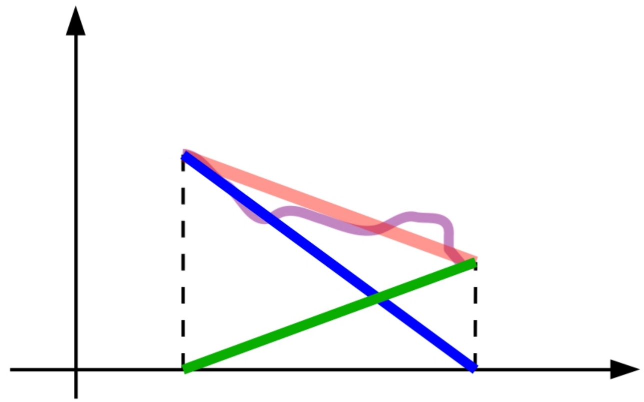
$$\int_0^L \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \frac{\partial}{\partial t}\left(\begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}\right) dx$$
$$- \int_0^L \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \frac{\partial}{\partial x}\left(\kappa(x)\frac{\partial}{\partial x}\left(\begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}\right)\right) dx - \int_0^L \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} s(x)\, dx = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

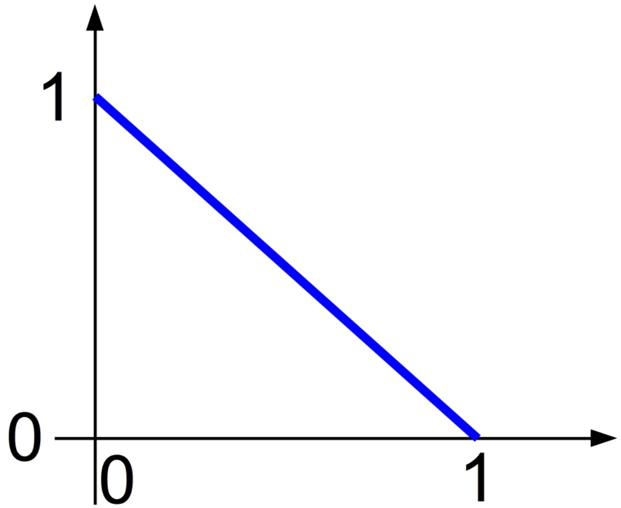
# Finite elements?

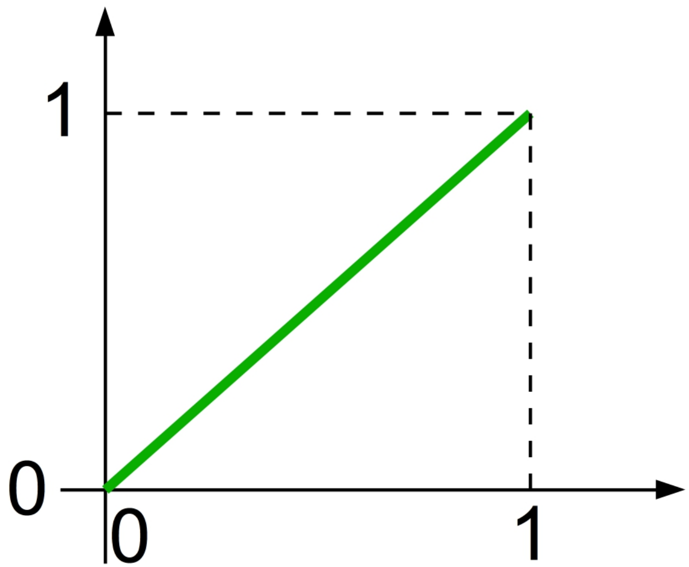


$$T(x) \approx N_1(x)T_1 + N_2(x)T_2$$

$$T(x) \approx \begin{bmatrix} N_1(x) & N_2(x) \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \mathbf{NT}$$






$$N_1(x) = 1 - \frac{x}{L}, \qquad N_2(x) = \frac{x}{L},$$

$$\int_0^L \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \frac{\partial}{\partial t} \left( \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \right) dx$$

$$- \int_0^L \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \frac{\partial}{\partial x} \left( \kappa(x) \frac{\partial}{\partial x} \left( \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \right) \right) dx - \int_0^L \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} s(x)\, dx = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

# Integration by part

$$\int_\Omega N_i \frac{\partial}{\partial x} \left( \kappa(x) \frac{\partial N_j}{\partial x} \right) dx = - \int_\Omega \kappa(x) \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} dx + \kappa(x) N_i \frac{\partial N_j}{\partial x} \Bigg|_{x_A}^{x_B}$$

# Local matrix

$$\begin{bmatrix} \int_0^L N_1 N_1\, dx & \int_0^L N_1 N_2\, dx \\ \int_0^L N_2 N_1\, dx & \int_0^L N_2 N_2\, dx \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$$

$$+ \bar{\kappa} \begin{bmatrix} \int_0^L \frac{\partial N_1}{\partial x} \frac{\partial N_1}{\partial x} dx & \int_0^L \frac{\partial N_1}{\partial x} \frac{\partial N_2}{\partial x} dx \\ \int_0^L \frac{\partial N_2}{\partial x} \frac{\partial N_1}{\partial x} dx & \int_0^L \frac{\partial N_2}{\partial x} \frac{\partial N_2}{\partial x} dx \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$$

$$- \bar{s} \begin{bmatrix} \int_0^L N_1\, dx \\ \int_0^L N_2\, dx \end{bmatrix} - \begin{bmatrix} N_1 q_A \Big|_{x_A} \\ N_2 q_B \Big|_{x_B} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{L}{3} & \frac{L}{6} \\ \frac{L}{6} & \frac{L}{3} \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} + \bar{\kappa} \begin{bmatrix} \frac{1}{L} & -\frac{1}{L} \\ -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} - \bar{s} \begin{bmatrix} \frac{L}{2} \\ \frac{L}{2} \end{bmatrix} - \begin{bmatrix} N_1 q_A \big|_{x_A} \\ N_2 q_B \big|_{x_B} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{M} \left( \frac{\partial \mathbf{T}}{\partial t} \right) + \mathbf{K}\,\mathbf{T} = \mathbf{F}$$

$$\mathbf{L}\,\mathbf{T}^{n+1} = \mathbf{R}\,\mathbf{T}^{n} + \mathbf{F}$$

$$\mathbf{L} = \begin{bmatrix} \frac{L}{3\Delta t} + \frac{\bar{\kappa}}{L} & \frac{L}{6\Delta t} - \frac{\bar{\kappa}}{L} \\ \frac{L}{6\Delta t} - \frac{\bar{\kappa}}{L} & \frac{L}{3\Delta t} + \frac{\bar{\kappa}}{L} \end{bmatrix} \qquad \mathbf{R} = \begin{bmatrix} \frac{L}{3\Delta t} & \frac{L}{6\Delta t} \\ \frac{L}{6\Delta t} & \frac{L}{3\Delta t} \end{bmatrix}$$

# Sum of local matrices -> global matrix

$$\begin{bmatrix} L_{11} & L_{12} & 0 & 0 & 0 \\ L_{21} & L_{22}+L_{11} & L_{12} & 0 & 0 \\ 0 & L_{21} & L_{22}+L_{11} & L_{12} & 0 \\ 0 & 0 & L_{21} & L_{22}+L_{11} & L_{12} \\ 0 & 0 & 0 & L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix}^{n+1}$$

$$= \begin{bmatrix} R_{11} & R_{12} & 0 & 0 & 0 \\ R_{21} & R_{22}+R_{11} & R_{12} & 0 & 0 \\ 0 & R_{21} & R_{22}+R_{11} & R_{12} & 0 \\ 0 & 0 & R_{21} & R_{22}+R_{11} & R_{12} \\ 0 & 0 & 0 & R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix}^{n}$$

$$+ sL \begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \\ 1 \\ \frac{1}{2} \end{bmatrix} + \begin{bmatrix} q_A \\ 0 \\ 0 \\ 0 \\ q_B \end{bmatrix}$$

# Global matrices

$$\mathbf{L}_G\,\mathbf{T}^{n+1} = \mathbf{R}_G\,\mathbf{T}^{n} + \mathbf{F}_G$$

$$\mathbf{L}_G\,\mathbf{T}^{n+1} = \mathbf{b}$$

$$\mathbf{T}^{n+1} = \mathbf{L}_G{}^{-1}\mathbf{b}$$

# Numerical integration

$$\int_a^b f(\xi)\, d\xi \approx \sum_{i=1}^{n} f(\xi_i) w_i$$

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| $n$ | $\xi_i$ | $w_i$ | $k$ |
|---|---|---|---|
| 1 | 0.0 | 2.0 | 1 |
| 2 | $\pm\sqrt{\frac{1}{3}}$ | 1.0 | 3 |

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$$\approx N_1\left(-\sqrt{1/3}\right) N_1\left(-\sqrt{1/3}\right) \times \frac{L}{2} \times 1.0$$

$$+ N_1\left(\sqrt{1/3}\right) N_1\left(\sqrt{1/3}\right) \times \frac{L}{2} \times 1.0$$

# Numerical integration

$$\int_a^b f(\xi)\, d\xi \approx \sum_{i=1}^{n} f(\xi_i) w_i$$

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|---|---|---|---|
| 1 | 0.0 | 2.0 | 1 |
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$$\begin{aligned}
\int_0^L N_1(x) N_1(x)\, dx &= \int_{-1}^{1} N_1(\xi) N_1(\xi) \frac{L}{2}\, d\xi \\
&\approx N_1\left(-\sqrt{1/3}\right) N_1\left(-\sqrt{1/3}\right) \times \frac{L}{2} \times 1.0 \\
&\quad + N_1\left(\sqrt{1/3}\right) N_1\left(\sqrt{1/3}\right) \times \frac{L}{2} \times 1.0 \\
&= \frac{L}{2}(0.6220084681 \times 1.0 + 0.04465819869 \times 1.0) \\
&= 0.333L,
\end{aligned}$$