Introduction to Finite Element Modelling in Geosciences:

Numerical Integration

Day 2

Patrick Sanan

(See Chapter 4 in the Course Notes)

The problem

For very simple cases, we can evaluate integrals exactly, when building our FE system:

$$\int_0^L N_1(x)N_2(x)dx = \int_0^L \left(1 - \frac{x}{L}\right)\left(\frac{x}{L}\right)dx = \frac{L}{6}$$

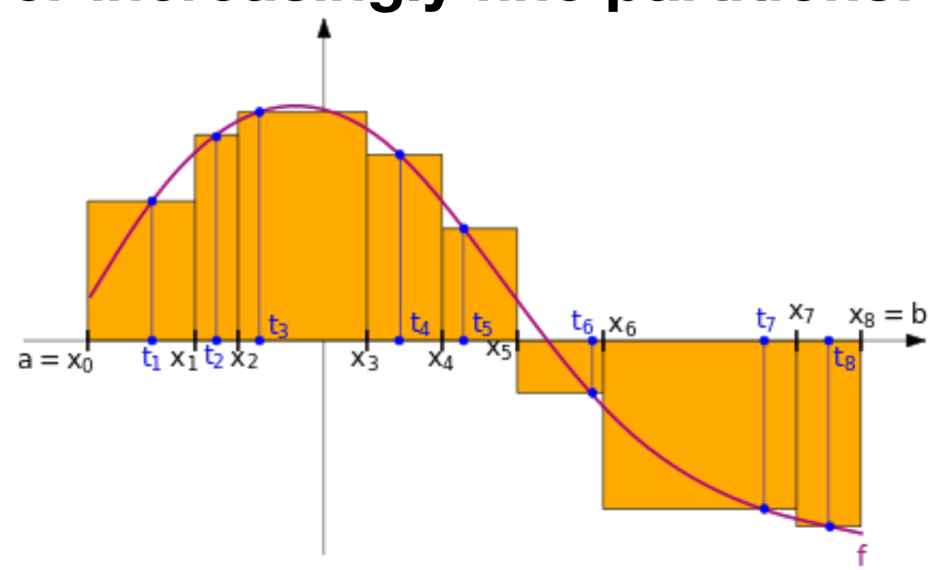
But for most cases of interest, we cannot:

$$\kappa(x) = \text{Some non-trivial function of } x$$

$$\int_0^L \kappa(x)\frac{\partial N_1(x)}{\partial x}\frac{\partial N_2(x)}{\partial x}dx = ?$$

The solution: numerical integration (quadrature)

Think back to learning about integrals. You can define them as the limit of finite sums over increasingly fine partitions:



https://de.wikipedia.org/wiki/Riemannsches_Integral#/media/File:Riemannsumme.svg

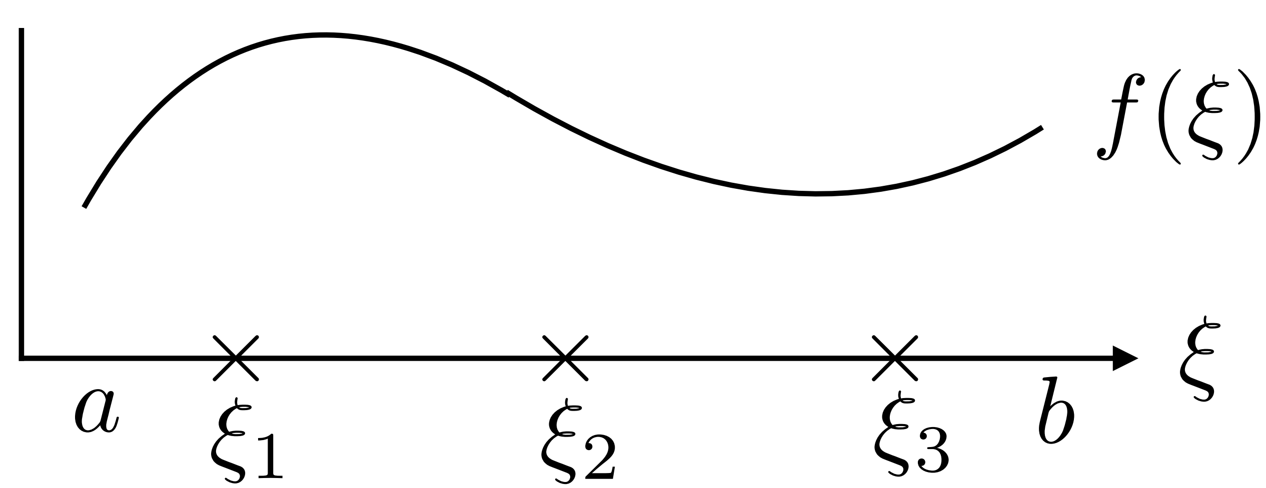
One can truncate this procedure, which works well for smooth functions

To increase accuracy, imagine using a higher-degree polynomial on each subinterval

Numerical Quadrature

Approximate an integral as a weighted sum of values at *quadrature points*

$$\int_a^b f(\xi)d\xi \approx \sum_{i=1}^{n} f(\xi_i)w_i$$



Numerical Quadrature

Integrate simple functions exactly

Higher-order quadrature rules can integrate higher-degree polynomials exactly

$$\int_a^b p(x)dx = \sum_{i=1}^{n} p(x_i)w_i$$

p **is a polynomial of degree** $\leq n$

Gauss-Legendre Quadrature

Use n quadrature points and weights chosen to integrate polynomials of degree less than 2n exactly over $[-1,1]$

$$\int_{-1}^{1} f(\xi)d\xi \approx \sum_{i=1}^{n} f(\xi_i)w_i$$

Values are tabulated for you in the notes (Table 4.1)

n	ξ_i	w_i	k
1	0.0	2.0	1
2	$\pm\sqrt{\frac{1}{3}}$	1.0	3
3	0.0	0.888888888888889	
	$\pm$0.774596669241483	0.555555555555556	5
4	$\pm$0.339981043584859	0.652145154862546	
	$\pm$0.861136311594053	0.347854845137454	7

Changes of Coordinates

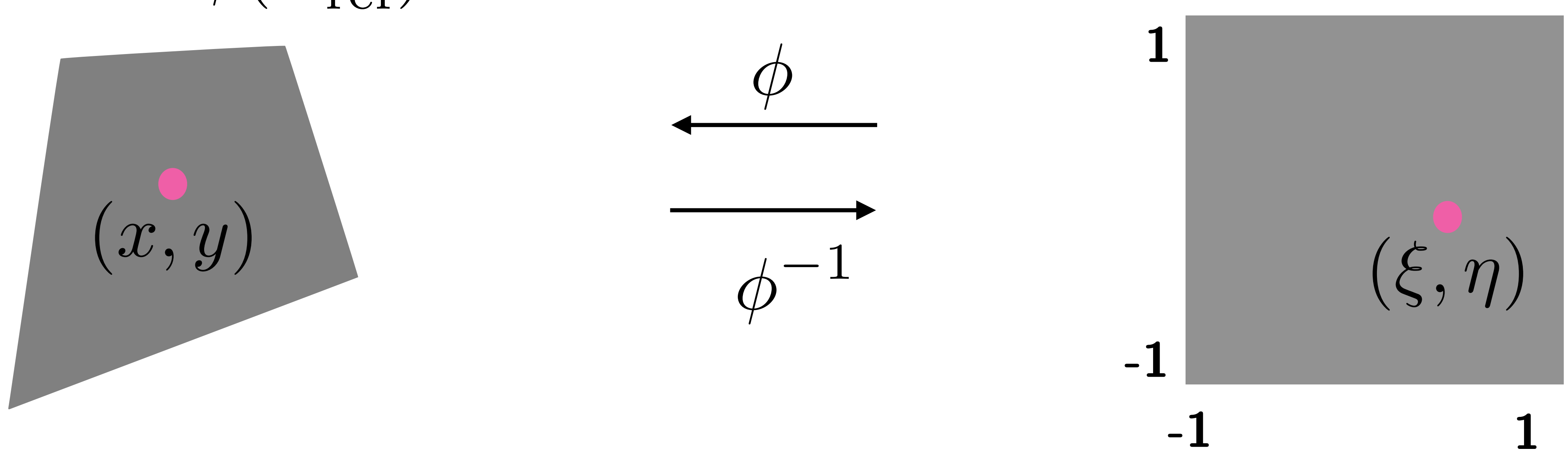
Our quadrature rules are tabulated to integrate over $[-1,1]$, but we'd like to integrate over general domains!

Spatial/Deformed Domain

$$\Omega^e = \phi(\Omega^e_{\text{ref}})$$

Reference Domain

$$\Omega^e_{\text{ref}}$$



These mappings let us think of the reference coordinates and spatial coordinates as functions of one another

$$\vec{x} \doteq (x, y, \ldots)$$

$$\vec{\xi} \doteq (\xi, \eta, \ldots)$$

$$\vec{x} = \phi(\vec{\xi})$$

The Jacobian

Note: we could (and usually do) write this instead of the more-accurate $\frac{\partial \phi_1}{\partial \xi}$

$$\vec{x} = \phi(\vec{\xi})$$

$$\vec{x} \doteq (x, y, \ldots)$$

$$\vec{\xi} \doteq (\xi, \eta, \ldots)$$

The Jacobian

$$J(\vec{\xi}) \doteq \left(D\phi(\vec{\xi})\right)^T = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \cdots \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

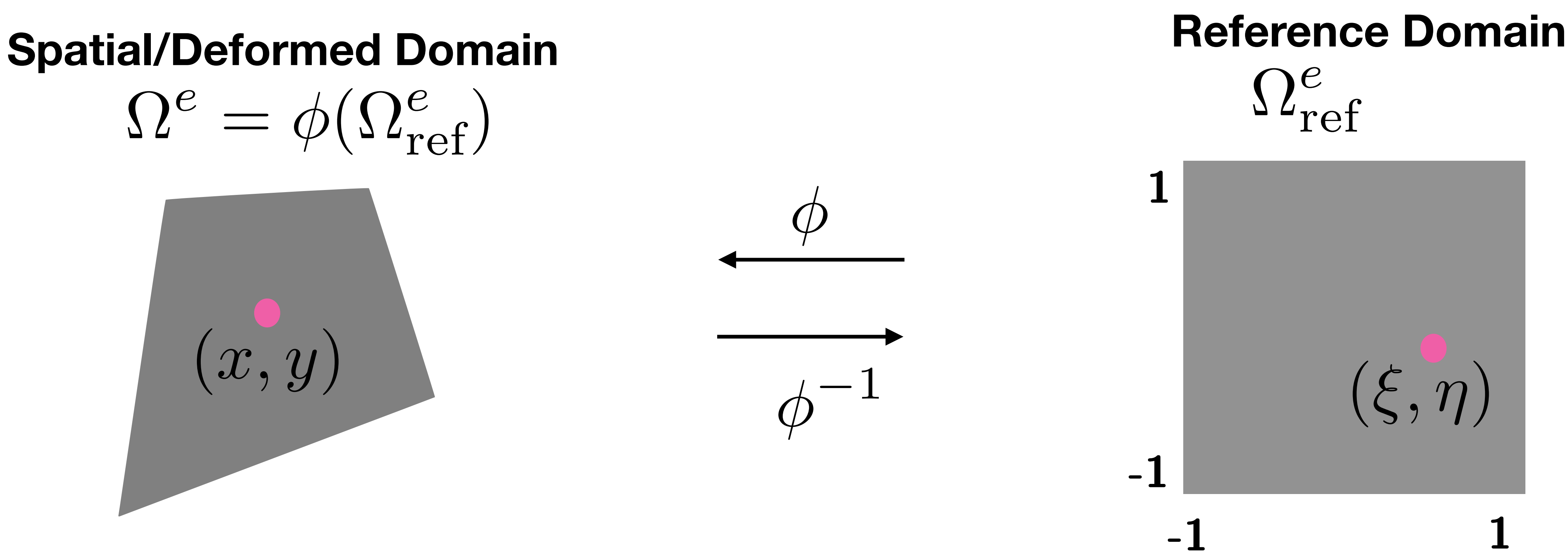
Chain Rule: $$\frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \xi} + \ldots$$

The Jacobian

$$\begin{bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \\ \vdots \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \cdots \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \vdots \end{bmatrix}$$

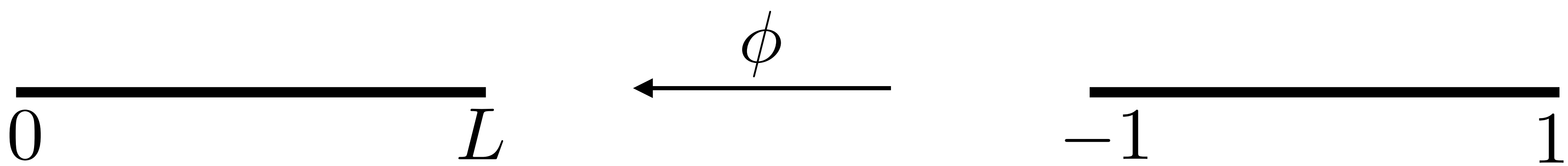
Terminology Warning! The literature is confusing. Sometimes "Jacobian" means the matrix above (which looks nice for transforming derivatives), sometimes it means the transpose of this matrix, and sometimes it means the determinant of the matrix above!

Change of Variables Formula



$$\int_{\Omega^e} f(\vec{x})d\vec{x} = \int_{\Omega^e_{\text{ref}}} f(\phi(\vec{\xi}))\,\det(J(\vec{\xi}))d\vec{\xi}$$

Change of Variables (1D)



$$x = \phi(\xi) = \frac{L}{2}(\xi + 1)$$

$$J(\xi) = \frac{\partial x}{\partial \xi} = \frac{L}{2} \qquad \det(J(\xi)) = \frac{L}{2}$$

Integrate a simple function:

$$\begin{aligned}
\int_0^L x^2 dx &= \int_{-1}^{1} (\phi(\xi))^2 \frac{L}{2} d\xi \\
&= \int_{-1}^{1} \left(\frac{L}{2}(\xi + 1)\right)^2 \frac{L}{2} d\xi \\
&= \left(\frac{L}{2}\right)^3 \int_{-1}^{1} (\xi + 1)^2 \\
&= \frac{L^3}{3}
\end{aligned}$$

Putting it together

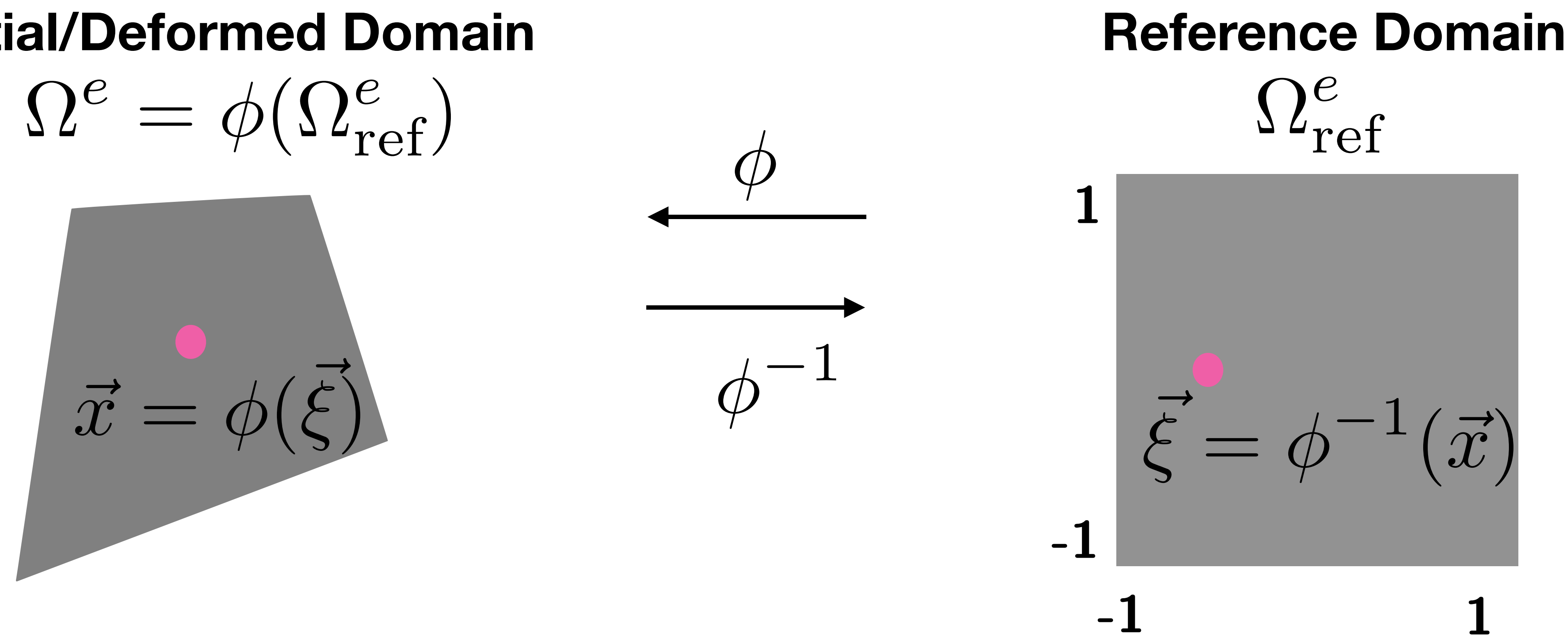
Now, you can numerically compute a 1D integral over an arbitrary domain, using Gauss-Legendre quadrature! (See exercise 4.2)

$$\int_a^b f(x)dx = \int_{-1}^1 f(\phi(\xi))\det(\frac{dx}{d\xi}(\xi))d\xi$$
$$\approx \sum_{i=1}^n f(\phi(\xi_i))\det(\frac{dx}{d\xi}(\xi_i))w_i$$

Note how this is just a weighted sum over quantities computable in the reference domain. This is critical for the efficiency of the FEM.

What about Derivatives?

Presentation of this topic can seem very confusing, because it's not usually made clear "what lives where"



Being very precise, the derivatives of these mappings map between the "tangent spaces" of the domains

What about Derivatives?

(Writing this precisely gets complex!)

$$\int_a^b \frac{df}{dx}(x)dx$$

$$= \int_a^b \frac{df}{d\xi}\frac{d\xi}{dx}dx$$

$$= \int_{-1}^1 \frac{df}{d\xi}(\phi(\xi))\frac{d\xi}{dx}(\phi(\xi))\,\det(\frac{dx}{d\xi}(\xi))d\xi$$

Note again that that these are all things you can evaluate in the reference domain!

The relevant example is (4.13) in the notes.