Introduction to Finite Element Modelling in Geosciences:

# From 1D to 2D

Day 3

Patrick Sanan

(See Chapter 5 in the Course Notes)

# From 1D to 2D

**The good news (and a powerful feature of the FEM) is that most things remain the same, regardless of dimension (or physics!)**

- Choose physics (PDE)
- Mesh domain
- Choose basis/interpolation functions
- Define element matrices and vectors
- Assemble elementwise matrices/vectors into global matrices/vectors
- Solve system
- Plot, postprocess, analyze, ...

**Main conceptual complication : Quadrature in 2D**
**(but if you understand the "change of variables" section from the last lecture, you are most of the way there)**

**Main practical complication: indexing**

# Numerical Quadrature in 2D

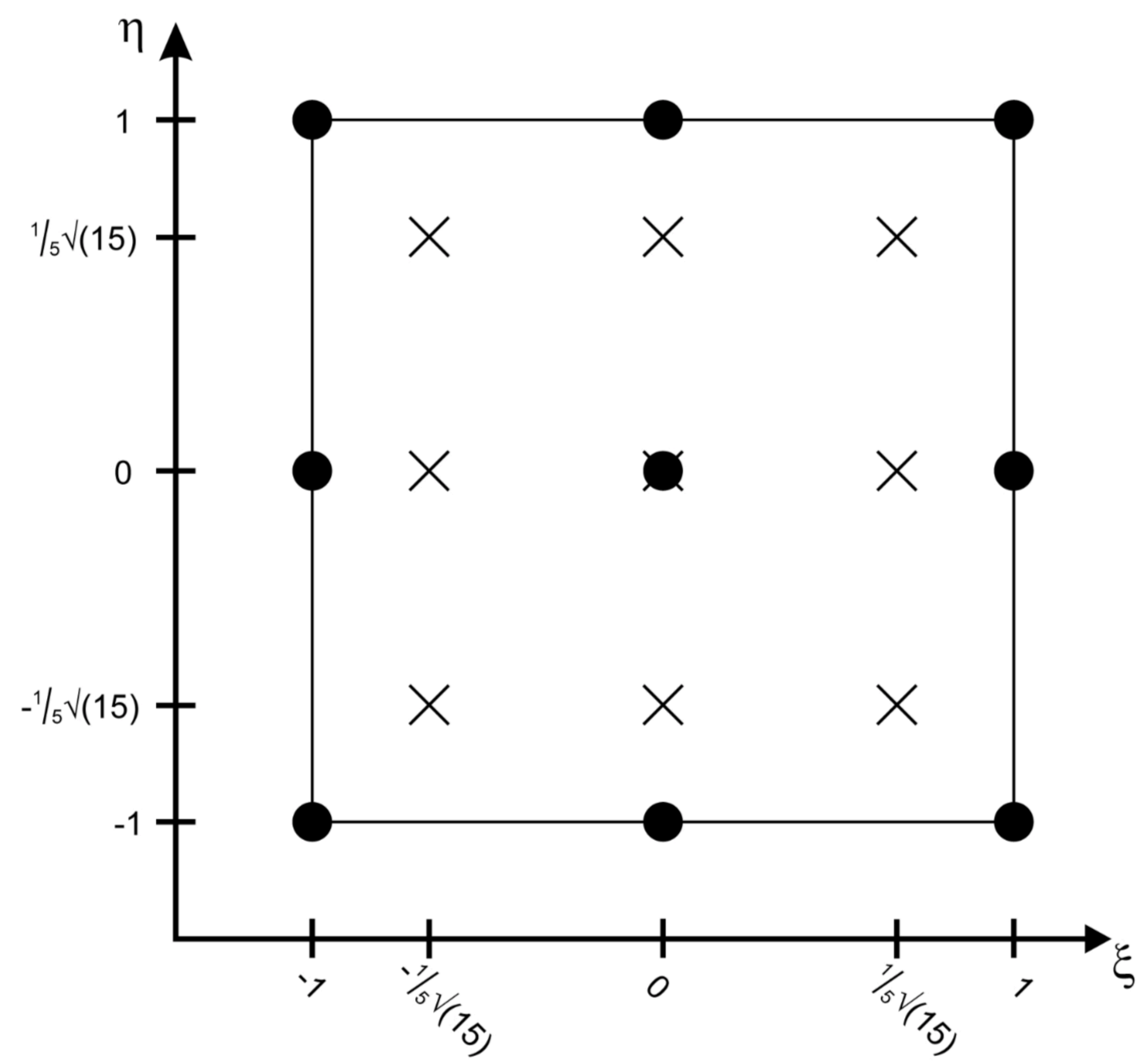
**2D Gauss-Legendre Integration - product of 1D rules!**

$$\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)d\vec{\xi} \approx \sum_{i=1}^{n_x}\sum_{j=1}^{n_y} f(\xi_i,\eta_j)w_i w_j$$

**Convenient to introduce linear numbering**

$$= \sum_{k=1}^{N_{\mathrm{ip}}} f(\xi_k,\eta_k)W_k$$

$$N_{\mathrm{ip}} \doteq n_x n_y$$



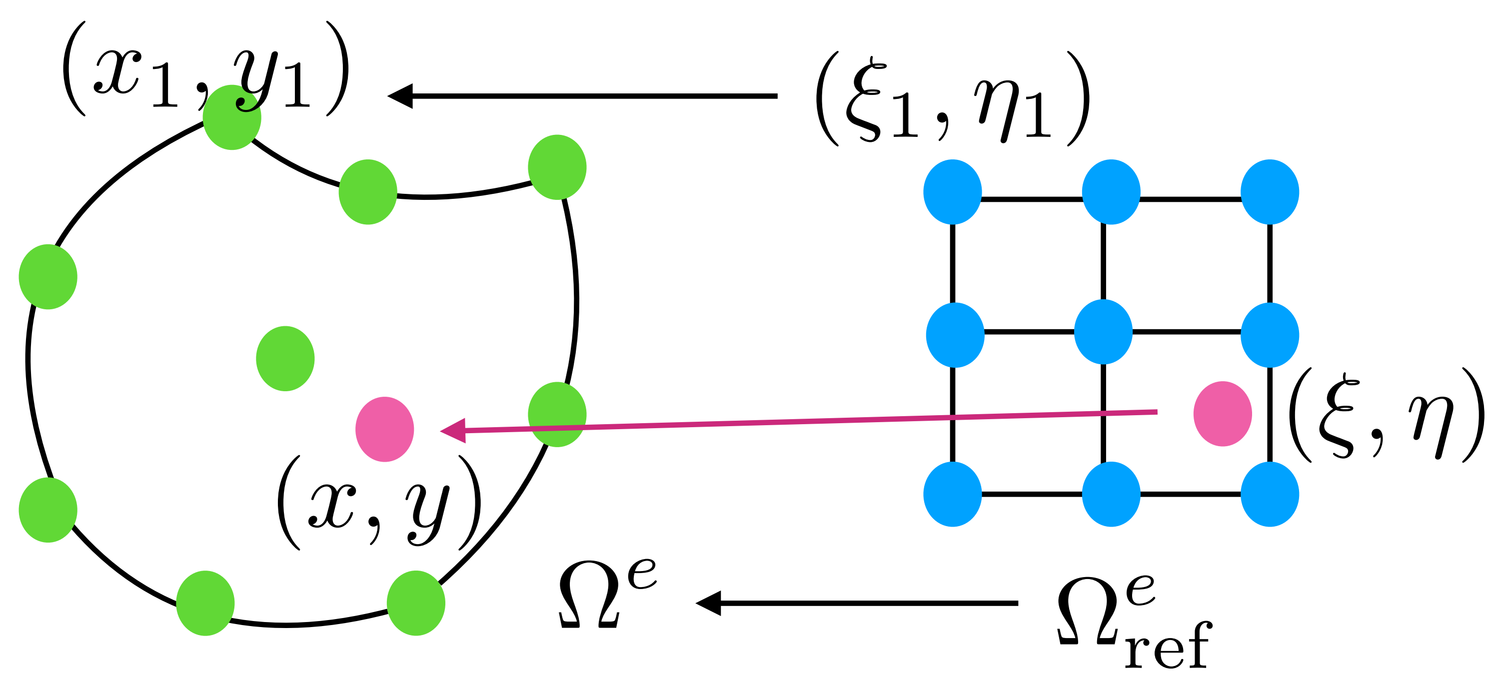

**Change of Coordinates - we've seen it before!**

**Warning! Make sure to correctly compute derivatives of shape functions (see 5.24)**

# Isoparametric Elements

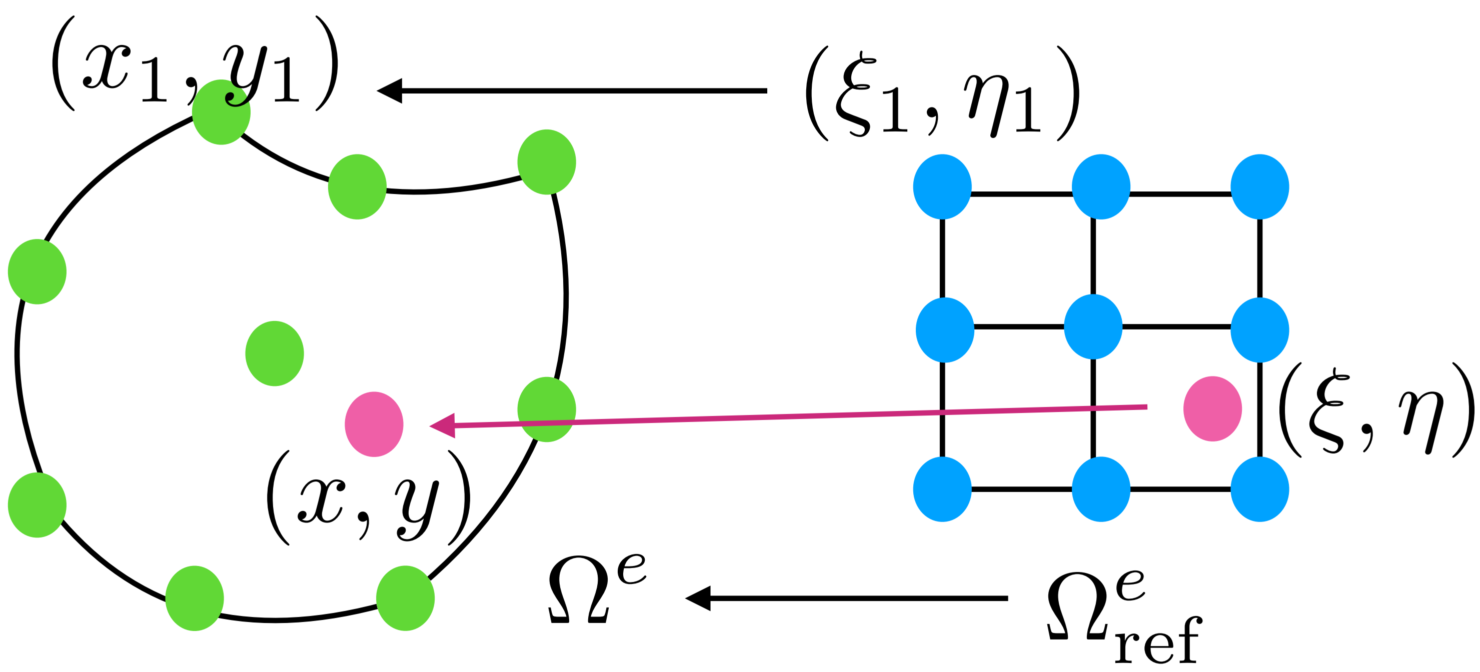
**Important point: you could choose any set of basis functions ( $N_i$ ) that you like, and if you use a reference element, you can choose any way you like to map it to the spatial/deformed element.**

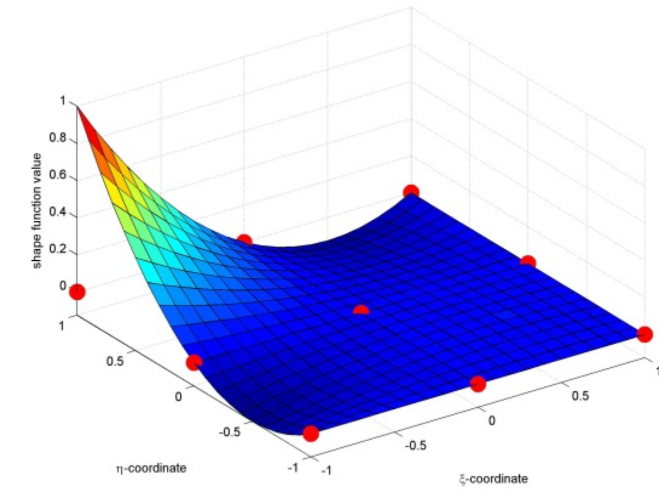
**We'd like to make choices that give, simple, efficient methods, though**




**Terminology Warning! Another place where FEM literature can be confusing is the term "element". Usually, we use this term to mean the subdomains into which we slice our domain. It can also be used to refer to the basis functions. Here, it refers to the latter!**

# Isoparametric Elements



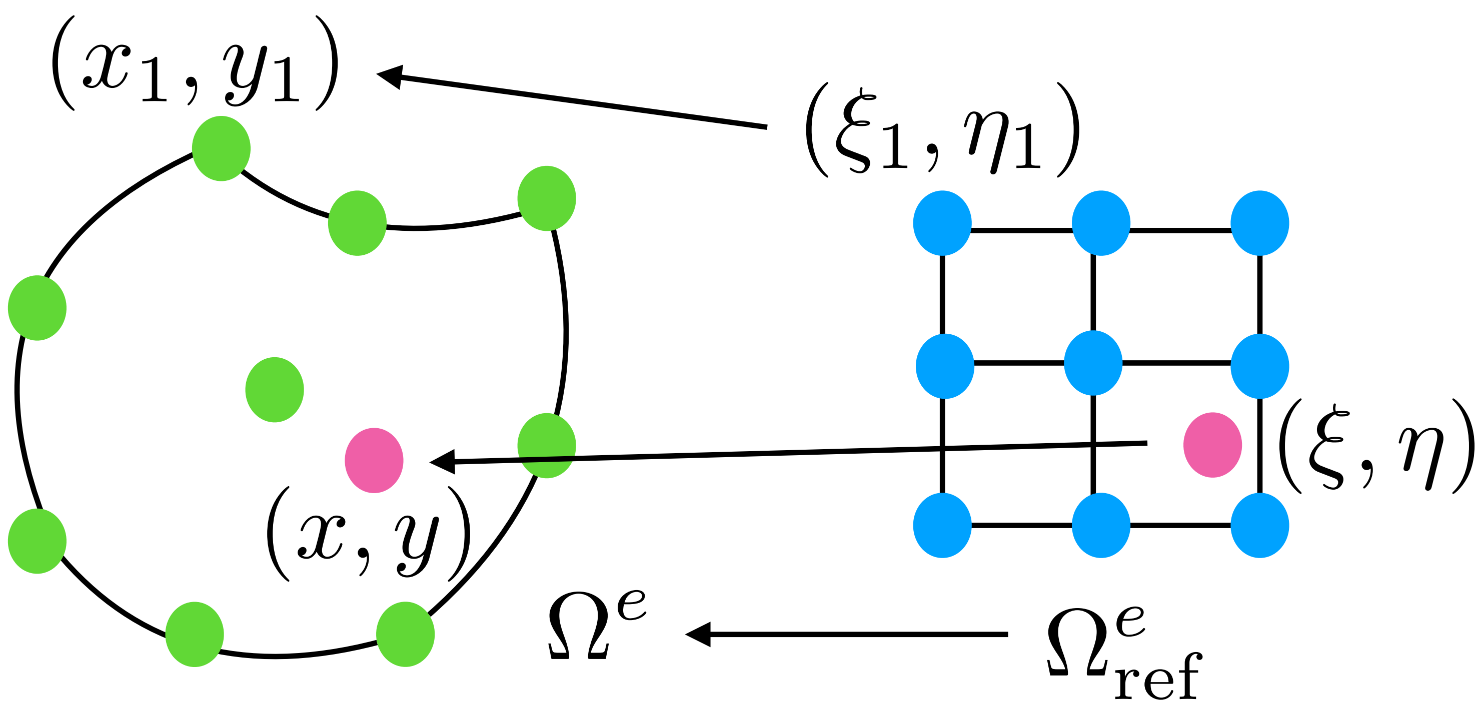

$N_1(\eta, \xi)$

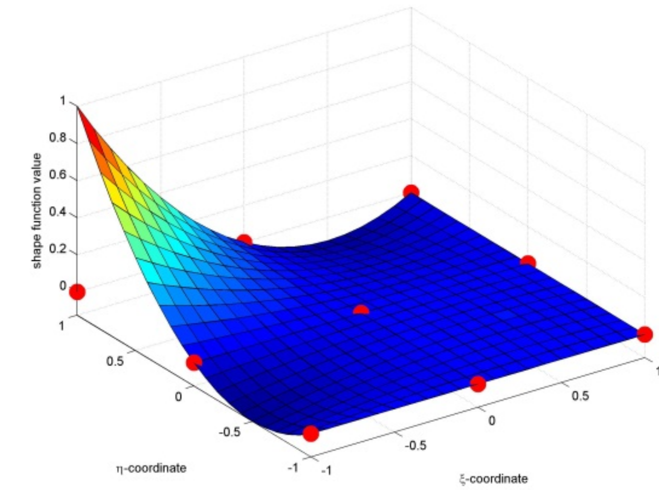
**Isoparametric elements use the same basis functions for the mapping from the reference element and for the interpolation functions!**

$$\vec{x} = (x, y, \ldots) = \phi(\vec{\xi})$$

$$\begin{bmatrix} x \\ y \\ \vdots \end{bmatrix} = \begin{bmatrix} N_1(\vec{\xi}) & N_2(\vec{\xi}) & N_3(\vec{\xi}) & \ldots \end{bmatrix} \begin{bmatrix} x_1 & y_1 & \ldots \\ x_2 & y_2 & \ldots \\ x_3 & y_3 & \ldots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

# Jacobian Computation



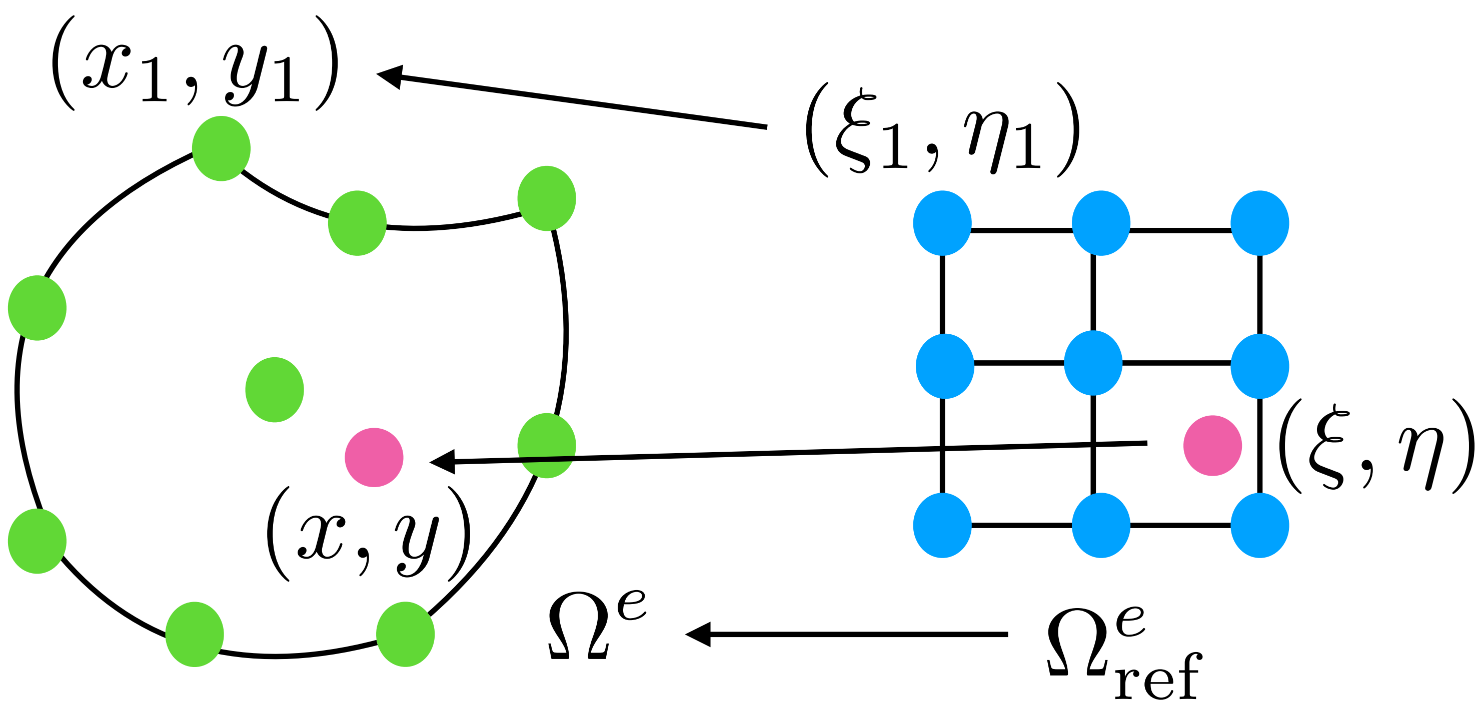

$N_1(\eta, \xi)$

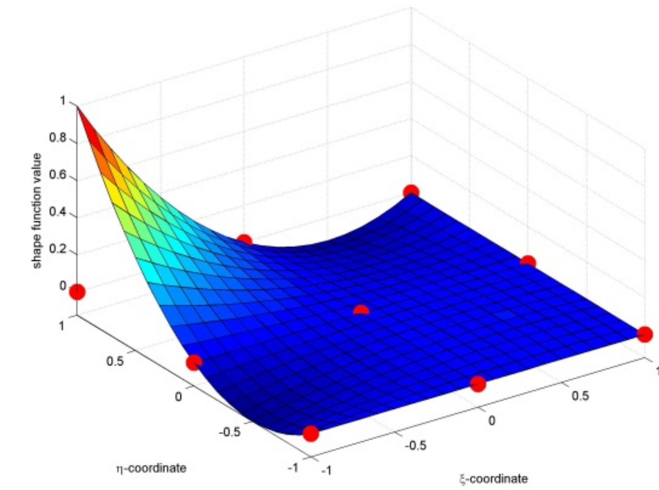
$$x = N_1(\xi,\eta)x_1 + N_2(\xi,\eta)x_2 + \dots$$
$$y = N_1(\xi,\eta)y_1 + N_2(\xi,\eta)y_2 + \dots$$

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

$$\mathbf{J} = \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \cdots & \frac{\partial N_9}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \cdots & \frac{\partial N_9}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ \vdots & \vdots \\ x_9 & y_9 \end{bmatrix}$$

# Jacobian Computation




$N_1(\eta, \xi)$

$$x = N_1(\xi,\eta)x_1 + N_2(\xi,\eta)x_2 + \dots$$
$$y = N_1(\xi,\eta)y_1 + N_2(\xi,\eta)y_2 + \dots$$

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

$$\mathbf{J} = \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \cdots & \frac{\partial N_9}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \cdots & \frac{\partial N_9}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ \vdots & \vdots \\ x_9 & y_9 \end{bmatrix}$$

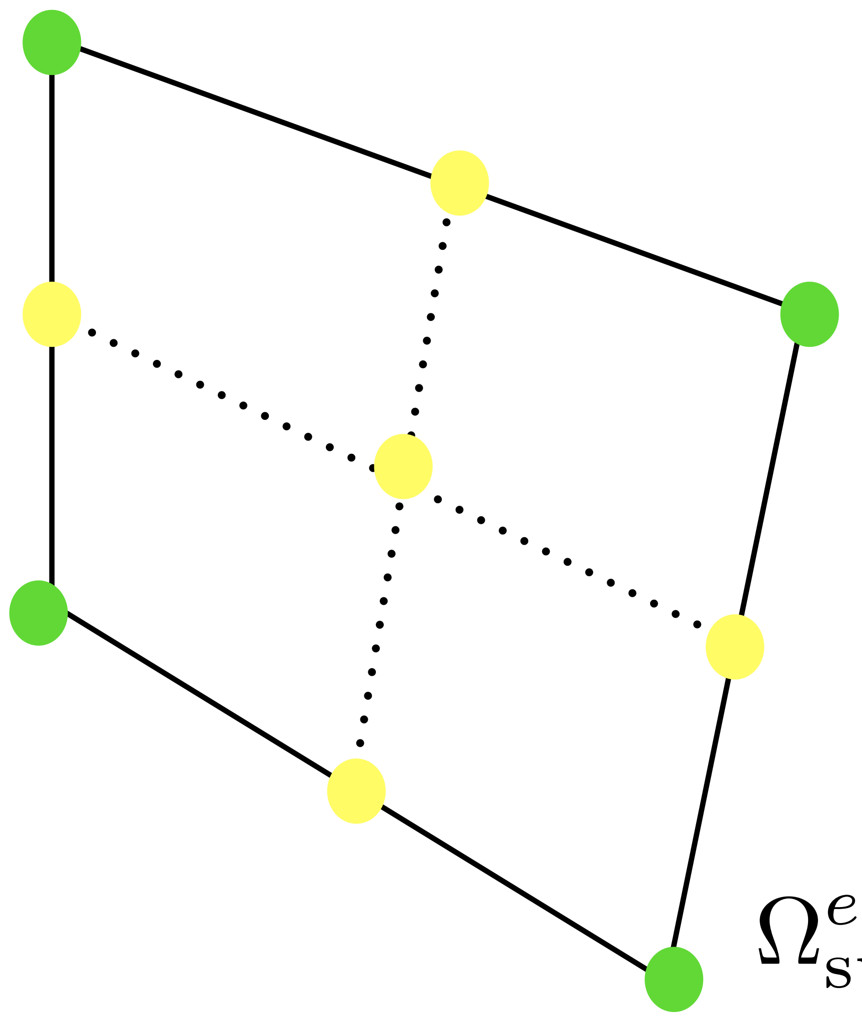
Can precompute this (once for each quadrature point)

This is different for each element

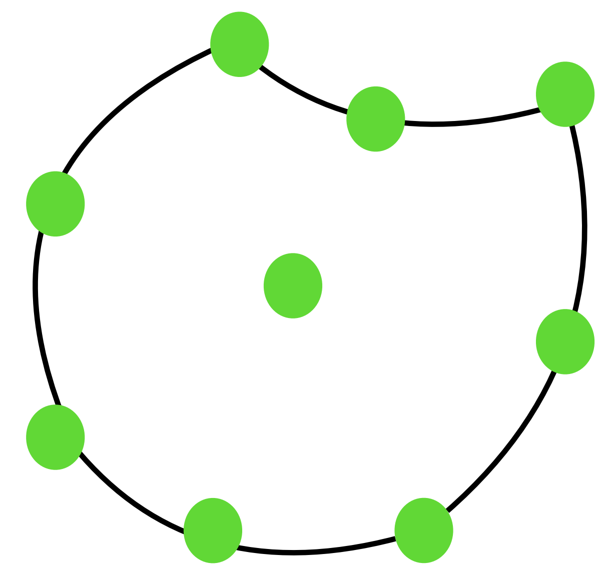
**Compare to (5.23) in the notes**

# What would a non-isoparametric element look like?

Yellow points defined by midpoints

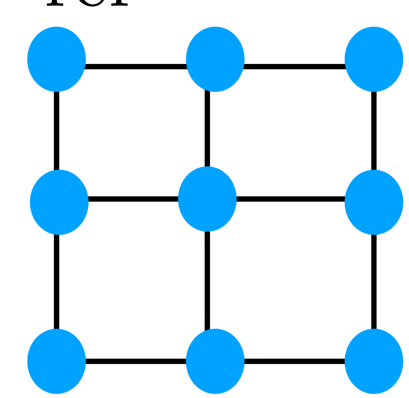


$\Omega^e_{\text{subparametric}}$



$\Omega^e_{\text{isoparametric}}$

$\Omega^e_{\text{ref}}$



**Defined by 4 green points, even though there are 9 basis functions**

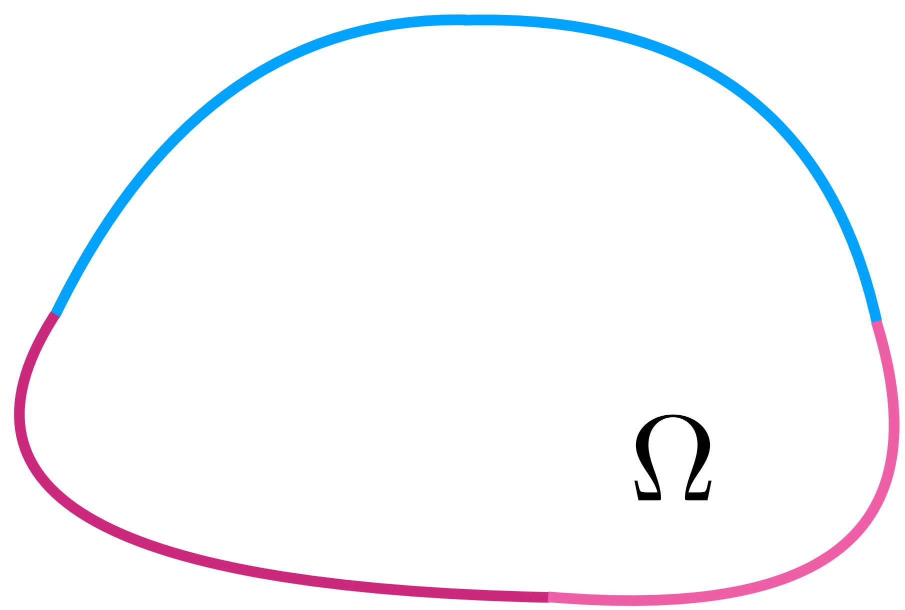
**4 < 9 , so <u>sub</u>parametric**

# Boundary Conditions

Dirichlet boundary conditions (prescribed value)

$$u = f(\vec{x})$$

**Modify system**




$$\frac{\partial u}{\partial n} = g(\vec{x})$$

**Modify righthand side**

Neumann boundary conditions (prescribed normal derivative)