Solving the elastic problem in Finite Elements

Non divergence of stresses:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0,$$

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Stress-strain relation:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{bmatrix}$$

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Strain-displacement relation:
(u is not a velocity!)

$$\begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

In matrix form:

$$\mathbf{B}^T \hat{\boldsymbol{\sigma}} = \mathbf{0},$$

$$\hat{\boldsymbol{\sigma}} = \mathbf{D}\hat{\boldsymbol{\epsilon}},$$

$$\hat{\boldsymbol{\epsilon}} = \mathbf{B}\mathbf{e},$$

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Differential operator

$$\mathbf{B} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}$$

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$$\mathbf{e} = \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

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Result:

$$\mathbf{B}^T\hat{\boldsymbol{\sigma}} = \mathbf{0}$$
$$\mathbf{B}^T\mathbf{D}\hat{\boldsymbol{\epsilon}} = \mathbf{0}$$
$$\mathbf{B}^T\mathbf{D}\mathbf{B}\mathbf{e} = \mathbf{0}$$

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Weak form?

$$\mathbf{B}^T \hat{\boldsymbol{\sigma}} = \mathbf{0}$$

$$\mathbf{B}^T \mathbf{D} \hat{\boldsymbol{\epsilon}} = \mathbf{0}$$

$$\mathbf{B}^T \mathbf{D} \mathbf{B} \mathbf{e} = \mathbf{0}$$

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Weak form?

$$u_x(x,y) \approx \begin{bmatrix} N_1(x,y) & N_2(x,y) & N_3(x,y) & N_4(x,y) \end{bmatrix} \begin{bmatrix} u_{x(1)} \\ u_{x(2)} \\ u_{x(3)} \\ u_{x(4)} \end{bmatrix} = \mathbf{N}\mathbf{u}_x$$

$$u_y(x,y) \approx \begin{bmatrix} N_1(x,y) & N_2(x,y) & N_3(x,y) & N_4(x,y) \end{bmatrix} \begin{bmatrix} u_{y(1)} \\ u_{y(2)} \\ u_{y(3)} \\ u_{y(4)} \end{bmatrix} = \mathbf{N}\mathbf{u}_y$$

Displacement vector in weak form:

$$\mathbf{r} = \begin{bmatrix} \mathbf{u}_x \\ \mathbf{u}_y \end{bmatrix} = \begin{bmatrix} u_{x(1)} \\ u_{x(2)} \\ u_{x(3)} \\ u_{x(4)} \\ u_{y(1)} \\ u_{y(2)} \\ u_{y(3)} \\ u_{y(4)} \end{bmatrix}$$

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Problem in weak form:

$$\mathbf{K}\,\mathbf{r} = \mathbf{F}$$

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where

$$\mathbf{K} = \int_{\Omega^e} \hat{\mathbf{B}}^T \mathbf{D} \hat{\mathbf{B}}\, dV,$$

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in weak form:

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Differential
operator
in weak form:

$$\hat{\mathbf{B}} = \begin{bmatrix} \frac{\partial \mathbf{N}}{\partial x} & 0 \\ 0 & \frac{\partial \mathbf{N}}{\partial y} \\ \frac{\partial \mathbf{N}}{\partial y} & \frac{\partial \mathbf{N}}{\partial x} \end{bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial x} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} & \frac{\partial N_4}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} & \frac{\partial N_4}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial x} \end{bmatrix}$$

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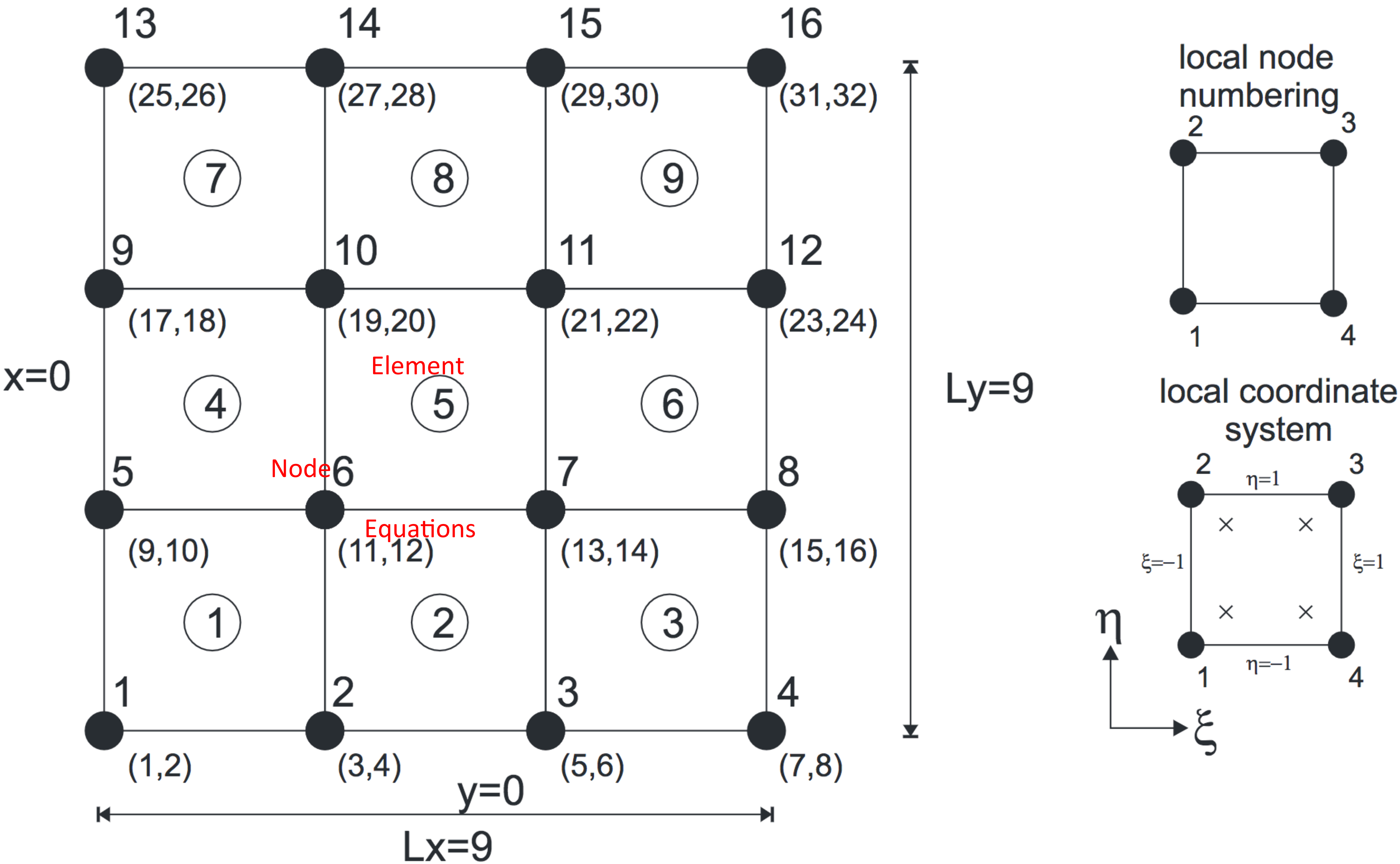
Forcing terms:

$$\mathbf{F} = \begin{bmatrix} -\oint_{\partial\Omega^e \cap \partial\Omega} \mathbf{N}^T \left(\sigma_{xx} n_x + \sigma_{xy} n_y\right) dS \\ -\oint_{\partial\Omega^e \cap \partial\Omega} \mathbf{N}^T \left(\sigma_{yx} n_x + \sigma_{yy} n_y\right) dS \end{bmatrix} = \begin{bmatrix} -\oint_{\partial\Omega^e \cap \partial\Omega} \mathbf{N}^T t_x\, dS \\ -\oint_{\partial\Omega^e \cap \partial\Omega} \mathbf{N}^T t_y\, dS \end{bmatrix}$$

Global node and element numbering



$$g\_num = \begin{bmatrix} 1 & 2 & 3 & 5 & 6 & 7 & 9 & 10 & 11 \\ 5 & 6 & 7 & 9 & 10 & 11 & 13 & 14 & 15 \\ 6 & 7 & 8 & 10 & 11 & 12 & 14 & 15 & 16 \\ 2 & 3 & 4 & 6 & 7 & 8 & 10 & 11 & 12 \end{bmatrix}$$

Relationship between elements and global node numbers

$$nf = \begin{bmatrix} 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 17 & 19 & 21 & 23 & 25 & 27 & 29 & 31 \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 & 22 & 24 & 26 & 28 & 30 & 32 \end{bmatrix}$$

Relationship between nodes and equation numbers

$$g\_g = \begin{bmatrix} 1 & 3 & 5 & 9 & 11 & 13 & 17 & 19 & 21 \\ 9 & 11 & 13 & 17 & 19 & 21 & 25 & 27 & 29 \\ 11 & 13 & 15 & 19 & 21 & 23 & 27 & 29 & 31 \\ 3 & 5 & 7 & 11 & 13 & 15 & 19 & 21 & 23 \\ 2 & 4 & 6 & 10 & 12 & 14 & 18 & 20 & 22 \\ 10 & 12 & 14 & 18 & 20 & 22 & 26 & 28 & 30 \\ 12 & 14 & 16 & 20 & 22 & 24 & 28 & 30 & 32 \\ 4 & 6 & 8 & 12 & 14 & 16 & 20 & 22 & 24 \end{bmatrix}$$

Relationship between elements and equation numbers