Finite Element Modelling for Geosciences: Code Verification and the Method of Manufactured Solutions

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The Problem

How do I know the code is working?

- As you've no doubt noticed, there are many, many ways to make mistakes in your implementation
- How do you show that the code "works"?
- One way we've seen is to confirm that our finite element solution looks "close" to an exact solution that we know beforehand
- But ..
  - What does "close" mean?
  - What if I don't have an exact solution to compare with?

Validation and Verification

- Verification
  - Is your code solving the equations correctly?
  - Does the code produce approximate solutions that converge at the correct rate?
  - The subject of Chapter 9 in the course notes
- Validation
  - Does the code produce physically-meaningful results?
  - Not covered in this course

Error estimates

- If we can construct an exact solution to the continuous problem, we can use it to verify our code
- The FEM is well-known to converge, if implemented correctly!
- Convergence is expressed in terms of the grid spacing and an error norm

$$||u - u^h||$$

- If an analytical expression is available, one can compute the solution at different spacings and examine how quickly it converges

$$||u - u^h|| \leq Ch^k$$

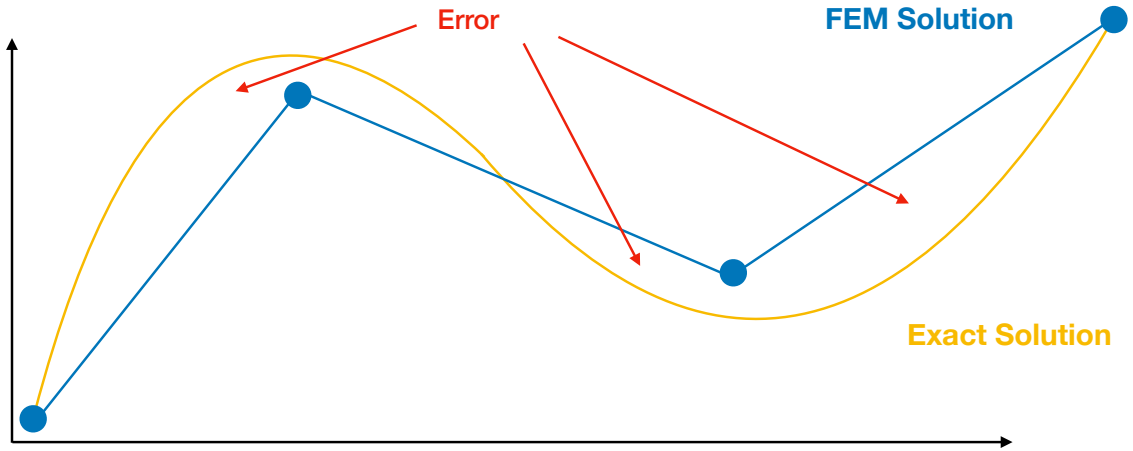
- The rate of convergence can be computed by fitting a straight line to a log-log plot of the grid spacing versus the error
- Obvious issue: what if we don't know u?

Computing errors

- The L^2 error

$$\left(\int_\Omega \left(u^h(x,y) - u(x,y) \right)^2 d\vec{x} \right)^{\frac{1}{2}}$$

- An integral over the entire domain; compute it with quadrature!



- Sometimes one sees "superconvergence" if only comparing nodal values.

```
err = sqrt(sum((T-T_exact).^2)) % Not an L^2 norm!
```

The Method of Manufactured Solutions (MMS)

- General approach - works for most PDE.
- Intuition: choose the solution you want and adjust the forcing and boundary terms to match.
- Choose ("manufacture") your solution
- Plug it into your equations (physics + material properties)
- Extract the required boundary and forcing terms (CAS tools like SAGE/SymPy/MAPLE/Mathematica/etc. are helpful)
- Now you can use your FEM code to compute an approximate solution, and compare.
- How to choose your manufactured solution
  - Make it complex enough
  - In particular, avoid low-order polynomials which might be exactly captured by your finite element basis functions
  - Good choices are combinations of sinusoidal or exponential functions

MMS Example

- Assume we're interested in approximate solutions to the Poisson equation (steady-state heat equation) on the unit square

$$\nabla \cdot (\kappa(x, y) \nabla T) = f(x, y), \quad \kappa(x, y) = 1 + x$$

- Note that we specified the coefficient structure, but *not* the boundary conditions or forcing
- Choose our own solution (depending on both x and y, not just a polynomial)

$$T_{\text{MMS}} \doteq \sin(x) cos(2y)$$

MMS Example

- Insert T_{MMS} into our equation[1] to obtain the forcing

$$f(x,y) = \frac{\partial}{\partial x}\left((1+x)\frac{\partial T_{\text{MMS}}}{\partial x}\right) + \frac{\partial}{\partial y}\left((1+x)\frac{\partial T_{\text{MMS}}}{\partial y}\right)$$

$$= \cos(2y)\left(\cos(x) - 5(x+1)\sin(x)\right)$$

- Use T_{MMS} to define boundary conditions, for example Dirichlet conditions

$$T(x,0) = T_{\text{MMS}}(x,0), \quad T(x,1) = T_{\text{MMS}}(x,1), \quad \forall x \in [0,1]$$

$$T(0,y) = T_{\text{MMS}}(0,y), \quad T(1,y) = T_{\text{MMS}}(1,y), \quad \forall y \in [0,1]$$

[1] I lazily just plugged this into Wolfram Alpha: `Div[(1+x)*Grad[Sin[x]*Cos[2*y]]]`

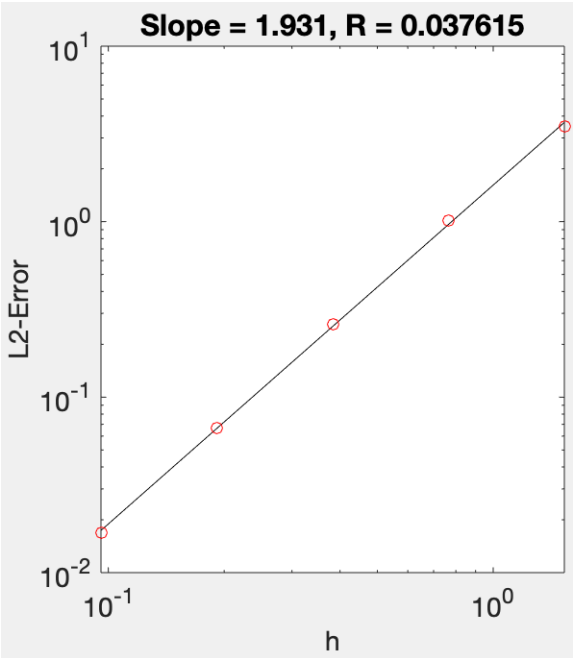
L2 error computation MATLAB code snippet

```
L2_error_2 = 0;
for i_el = 1:el_tot
    nodes_el = element_node_map(:,i_el);
    coord_el = gcoord(:,nodes_el)';
    T_el = T(nodes_el);

    for i_pt = 1:pts_per_el
        jac      = dNdxi(:,:,i_pt) * coord_el;
        det_jac = det(jac)

        coord_pt = N(i_pt,:) * coord_el;
        T_mms_pt = T_MMS(coord_pt(1), coord_pt(2));
        T_fe_pt  = N(i_pt,:) * T_el;

        L2_error_2 = L2_error_2 + \
          (T_mms_pt - T_fe_pt)^2 * det_jac * w(i_pt);
    end
L2_error = sqrt(L2_error_2)
```

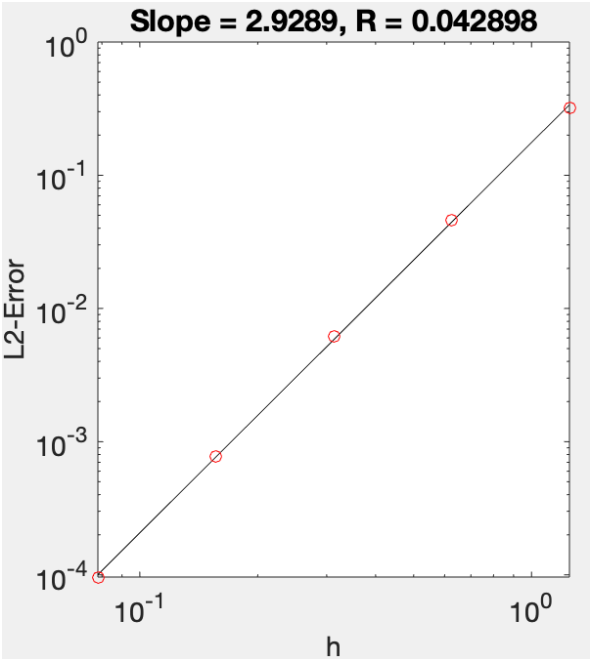
- The expected convergence rate, from the course notes, is

$$\left[\int_\Omega (T - T^h)^2 d\vec{x}\right]^{\frac{1}{2}} \leq Ch^{p+1}$$

- Our modified code shows slopes of approximately 2 and 3 on a log-log plot, so we are now much more confident that the implementation is correct.



Bilinear elements, $p = 1$



Biquadratic elements, $p = 2$

MMS: Wrinkle - non-physical forcing

- If using MMS for something like the Stokes equations (Friday), you will perhaps run into the following issue
- As an example, let's say that I have written a FE code to solve this (very) simple PDE:

$$\frac{d}{dx}\left(k(x)\frac{du}{dx}\right) = 0, \quad u \in [0,1], \quad u(0) = 1, u(1) = 2$$

- What issue might I run into if I apply the MMS?
- I may have written my code assuming zero forcing!
- So I may have to **modify my weak form** to take the additional forcing into account. Don't forget to consider terms like

$$\mathbf{F} = \int_{\Omega^e} \mathbf{N}^T f \; d\vec{x}$$

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