Finite Element Modelling in Geosciences

FEM in 2D for viscous materials

Introduction to Finite Element Modelling in Geosciences

Summer 2019

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Recap from 2D continuum mechanics and rheology

Equations	Matrix notation	Short form

Incompressibility

$$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}=0$$

$$\begin{bmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y}\end{bmatrix}\begin{Bmatrix}v_x\\ v_y\end{Bmatrix}=0$$

$$\nabla\tilde{\mathbf{v}}=0$$

Conservation of linear momentum (i.e., force balance equation)

$$\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{xy}}{\partial y}+F_x=0$$

$$\frac{\partial \sigma_{xy}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}+F_y=0$$

$$\begin{bmatrix}\frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y}\\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x}\end{bmatrix}\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{Bmatrix}+\begin{Bmatrix}F_x\\ F_x\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix}$$

$$\tilde{\mathbf{B}}^T\tilde{\boldsymbol{\sigma}}+\tilde{\mathbf{F}}=\mathbf{0}$$

Rheology

$$\sigma_{xx}=4\eta/3\,\dot{\varepsilon}_{xx}-2\eta/3\,\dot{\varepsilon}_{yy}-\tilde{p}$$

$$\sigma_{yy}=-2\eta/3\,\dot{\varepsilon}_{xx}+4\eta/3\,\dot{\varepsilon}_{yy}-\tilde{p}$$

$$\sigma_{xy}=\eta\dot{\gamma}_{xy}$$

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{Bmatrix}=\begin{bmatrix}4\eta/3 & -2\eta/3 & 0\\ -2\eta/3 & 4\eta/3 & 0\\ 0 & 0 & \eta\end{bmatrix}\begin{Bmatrix}\dot{\varepsilon}_{xx}\\ \dot{\varepsilon}_{yy}\\ \dot{\gamma}_{xy}\end{Bmatrix}-\begin{Bmatrix}1\\ 1\\ 0\end{Bmatrix}\tilde{p}$$

$$\tilde{\boldsymbol{\sigma}}=\mathbf{D}\tilde{\dot{\boldsymbol{\varepsilon}}}-\mathbf{m}\tilde{p}$$

Kinematic relation

$$\dot{\varepsilon}_{xx}=\partial v_x/\partial x$$

$$\dot{\varepsilon}_{yy}=\partial v_y/\partial y$$

$$\dot{\gamma}_{xy}=\partial v_x/\partial y+\partial v_y/\partial x$$

$$\begin{Bmatrix}\dot{\varepsilon}_{xx}\\ \dot{\varepsilon}_{yy}\\ \dot{\gamma}_{xy}\end{Bmatrix}=\begin{bmatrix}\partial/\partial x & 0\\ 0 & \partial/\partial y\\ \partial/\partial y & \partial/\partial x\end{bmatrix}\begin{Bmatrix}v_x\\ v_y\end{Bmatrix}$$

$$\tilde{\dot{\boldsymbol{\varepsilon}}}=\tilde{\mathbf{B}}\tilde{\mathbf{v}}$$

Total system of equations

- Incompressibility

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

! NEW !

- Short form

$$\tilde{\mathbf{B}}^T \mathbf{D} \tilde{\mathbf{B}} \tilde{\mathbf{v}} - \nabla^T \tilde{p} + \tilde{\mathbf{F}} = \mathbf{0}$$

- Matrix notation

$$\begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} \frac{4}{3}\eta & -\frac{2}{3}\eta & 0 \\ -\frac{2}{3}\eta & \frac{4}{3}\eta & 0 \\ 0 & 0 & \eta \end{bmatrix} \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix} \begin{Bmatrix} v_x \\ v_y \end{Bmatrix} - \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} p + \begin{Bmatrix} F_x \\ F_y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

- Written out (2 equations)

$$\frac{\partial}{\partial x}\left[\frac{4}{3}\eta\frac{\partial v_x}{\partial x} - \frac{2}{3}\eta\frac{\partial v_y}{\partial y}\right] + \frac{\partial}{\partial y}\left[\eta\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)\right] - \frac{\partial p}{\partial x} + F_x = 0$$

$$\frac{\partial}{\partial y}\left[-\frac{2}{3}\eta\frac{\partial v_x}{\partial x} + \frac{4}{3}\eta\frac{\partial v_y}{\partial y}\right] + \frac{\partial}{\partial x}\left[\eta\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)\right] - \frac{\partial p}{\partial y} + F_y = 0$$

Force balance equation: Deriving the weak form

- Test functions: $\mathbf{N} = \begin{bmatrix} N_1 & \cdots & N_9 \end{bmatrix}$

① Multiplication with test functions:

$$\mathbf{N}^T \frac{\partial}{\partial x}\left[\frac{4}{3}\eta\frac{\partial v_x}{\partial x} - \frac{2}{3}\eta\frac{\partial v_y}{\partial y}\right] + \mathbf{N}^T \frac{\partial}{\partial y}\left[\eta\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)\right] - \mathbf{N}^T \frac{\partial p}{\partial x} + \mathbf{N}^T F_x = 0$$

$$\mathbf{N}^T \frac{\partial}{\partial y}\left[-\frac{2}{3}\eta\frac{\partial v_x}{\partial x} + \frac{4}{3}\eta\frac{\partial v_y}{\partial y}\right] + \mathbf{N}^T \frac{\partial}{\partial x}\left[\eta\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)\right] - \mathbf{N}^T \frac{\partial p}{\partial y} + \mathbf{N}^T F_y = 0$$

② Spatial integration

$$\int_0^{\Delta x}\int_0^{\Delta y} \mathbf{N}^T \frac{\partial}{\partial x}\left[\frac{4}{3}\eta\frac{\partial v_x}{\partial x} - \frac{2}{3}\eta\frac{\partial v_y}{\partial y}\right] \mathrm{d}x\mathrm{d}y + \int_0^{\Delta x}\int_0^{\Delta y} \mathbf{N}^T \frac{\partial}{\partial y}\left[\eta\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)\right] \mathrm{d}x\mathrm{d}y - \int_0^{\Delta x}\int_0^{\Delta y} \mathbf{N}^T \frac{\partial p}{\partial x} \mathrm{d}x\mathrm{d}y + \int_0^{\Delta x}\int_0^{\Delta y} \mathbf{N}^T F_x \mathrm{d}x\mathrm{d}y = 0$$

$$\int_0^{\Delta x}\int_0^{\Delta y} \mathbf{N}^T \frac{\partial}{\partial y}\left[-\frac{2}{3}\eta\frac{\partial v_x}{\partial x} + \frac{4}{3}\eta\frac{\partial v_y}{\partial y}\right] \mathrm{d}x\mathrm{d}y + \int_0^{\Delta x}\int_0^{\Delta y} \mathbf{N}^T \frac{\partial}{\partial x}\left[\eta\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)\right] \mathrm{d}x\mathrm{d}y - \int_0^{\Delta x}\int_0^{\Delta y} \mathbf{N}^T \frac{\partial p}{\partial y} \mathrm{d}x\mathrm{d}y + \int_0^{\Delta x}\int_0^{\Delta y} \mathbf{N}^T F_y \mathrm{d}x\mathrm{d}y = 0$$

Force balance equation: Deriving the weak form

③ Integration by parts and dropping arising boundary terms

$$\int_0^{\Delta x}\int_0^{\Delta y} \frac{\partial \mathbf{N}^T}{\partial x}\left[\frac{4}{3}\eta\frac{\partial v_x}{\partial x}-\frac{2}{3}\eta\frac{\partial v_y}{\partial y}\right]\mathrm{d}y\mathrm{d}x+\int_0^{\Delta y}\int_0^{\Delta x}\frac{\partial \mathbf{N}^T}{\partial y}\eta\left(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}\right)\mathrm{d}x\mathrm{d}y-\int_0^{\Delta x}\int_0^{\Delta y}\frac{\partial \mathbf{N}^T}{\partial x}p\,\mathrm{d}x\mathrm{d}y=\int_0^{\Delta x}\int_0^{\Delta y}\mathbf{N}^T F_x\mathrm{d}x\mathrm{d}y$$

$$\int_0^{\Delta y}\int_0^{\Delta x} \frac{\partial \mathbf{N}^T}{\partial y}\left[-\frac{2}{3}\eta\frac{\partial v_x}{\partial x}+\frac{4}{3}\eta\frac{\partial v_y}{\partial y}\right]\mathrm{d}x\mathrm{d}y+\int_0^{\Delta x}\int_0^{\Delta y}\frac{\partial \mathbf{N}^T}{\partial x}\eta\left(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}\right)\mathrm{d}y\mathrm{d}x-\int_0^{\Delta x}\int_0^{\Delta y}\frac{\partial \mathbf{N}^T}{\partial y}p\,\mathrm{d}x\mathrm{d}y=\int_0^{\Delta x}\int_0^{\Delta y}\mathbf{N}^T F_y\mathrm{d}x\mathrm{d}y$$

This is the weak form of the force balance equation!

The finite element approximation

2D elasticity

$$u_x(x,y) \approx N_1 u_{x,1} + N_2 u_{x,2} + N_3 u_{x,3} + N_4 u_{x,4}$$

$$= \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{Bmatrix} u_{x,1} \\ u_{x,2} \\ u_{x,3} \\ u_{x,4} \end{Bmatrix} = \mathbf{N}\mathbf{u}_x$$

$$u_y(x,y) \approx \mathbf{N}\mathbf{u}_y$$

Requirements for the shape functions:

- Equal 1 at one nodal point
- Equal 0 at all other nodal points
- Sum of all shape functions has to be 1 in the whole finite element.

2D viscosity

$$v_x(x,y) \approx N_1 v_{x,1} + N_2 v_{x,2} + \ldots + N_9 v_{x,9}$$

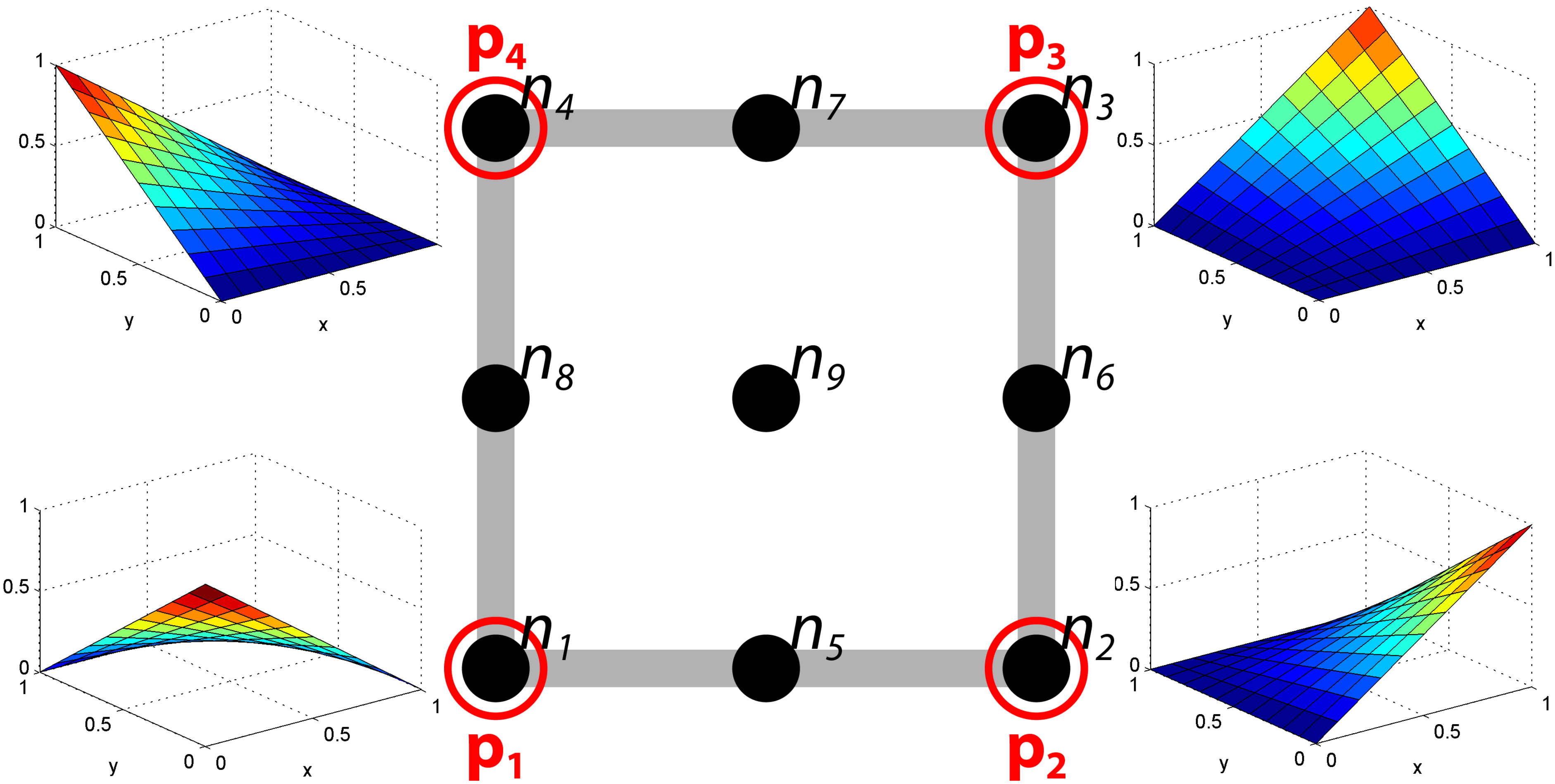
$$= \begin{bmatrix} N_1 & N_2 & \ldots & N_9 \end{bmatrix} \begin{Bmatrix} v_{x,1} \\ v_{x,2} \\ \vdots \\ v_{x,9} \end{Bmatrix} = \mathbf{N}\mathbf{v}_x$$

$$v_y(x,y) \approx \mathbf{N}\mathbf{v}_y$$

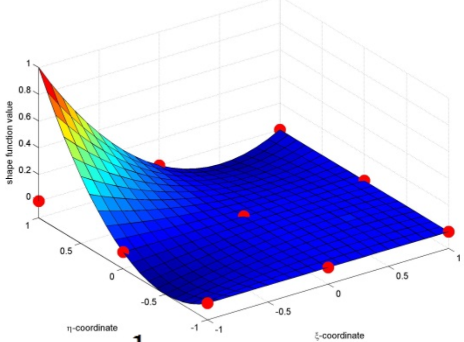
$$p(x,y) \approx N_1^p p_1 + N_2^p p_2 + N_3^p p_3 + N_4^p p_4$$

- 9 bi-quadratic shape functions for velocity
- 4 bi-linear shape functions for pressure (the same as for 2D elasticity)

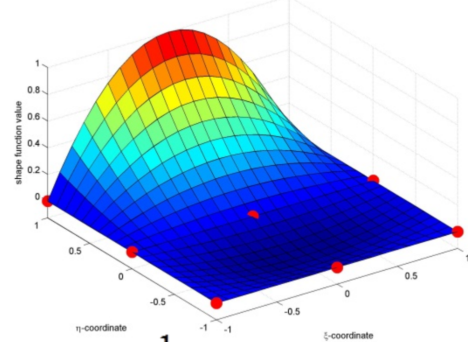
The Q2Q1-element: bi-linear pressure shape functions



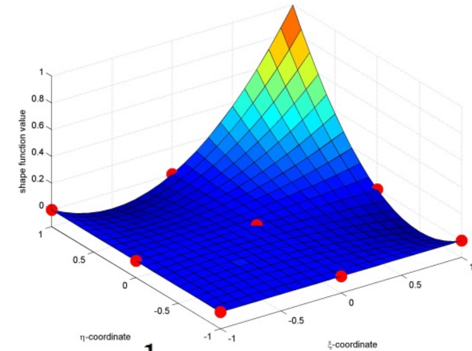
The Q2Q1-element: bi-quadratic velocity shape functions



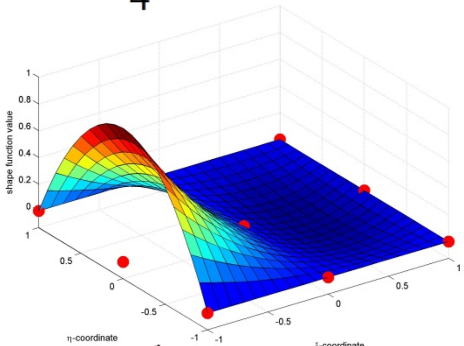
$$N_4 = \frac{1}{4}\left(\xi^2 - \xi\right)\left(\eta^2 + \eta\right)$$



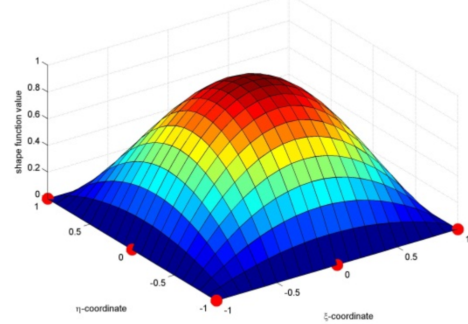
$$N_7 = -\frac{1}{2}\left(\xi^2 - 1\right)\left(\eta^2 + \eta\right)$$



$$N_3 = \frac{1}{4}\left(\xi^2 + \xi\right)\left(\eta^2 + \eta\right)$$

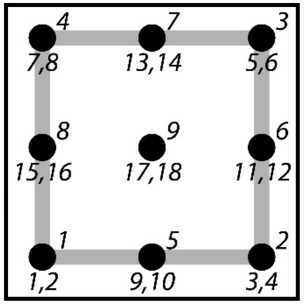


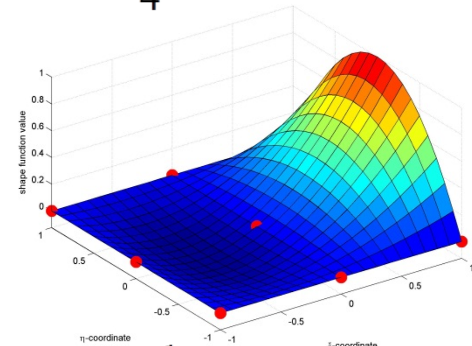
$$N_8 = -\frac{1}{2}\left(\xi^2 - \xi\right)\left(\eta^2 - 1\right)$$



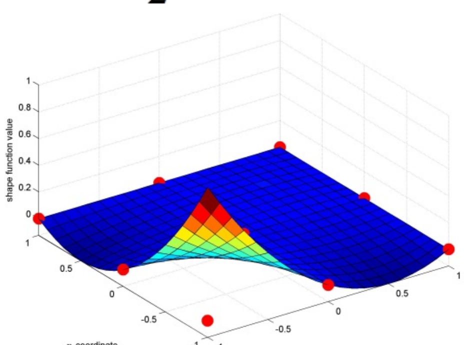
$$N_9 = \left(\xi^2 - 1\right)\left(\eta^2 - 1\right)$$

Element for velocity

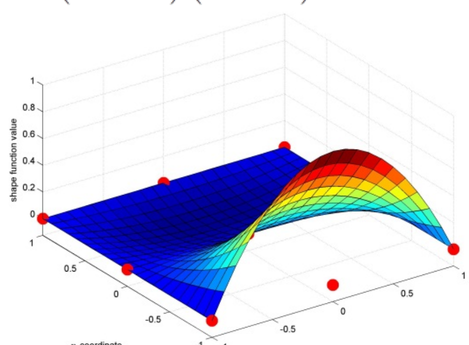




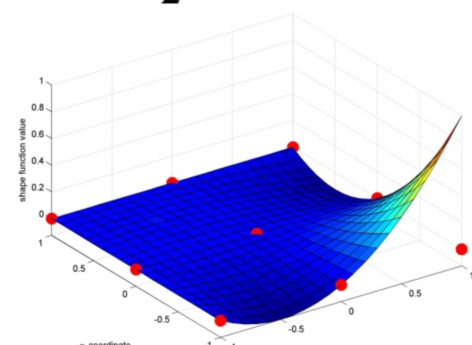
$$N_6 = -\frac{1}{2}\left(\xi^2 + \xi\right)\left(\eta^2 - 1\right)$$



$$N_1 = \frac{1}{4}\left(\xi^2 - \xi\right)\left(\eta^2 - \eta\right)$$



$$N_5 = -\frac{1}{2}\left(\xi^2 - 1\right)\left(\eta^2 - \eta\right)$$



$$N_2 = \frac{1}{4}\left(\xi^2 + \xi\right)\left(\eta^2 - \eta\right)$$

Force balance equations: The FE-approximation

④ Apply finite element approximation

$$\int_0^{\Delta x}\int_0^{\Delta y}\frac{\partial \mathbf{N}^T}{\partial x}\left[\frac{4}{3}\eta\frac{\partial \mathbf{N}}{\partial x}\mathbf{v}_x-\frac{2}{3}\eta\frac{\partial \mathbf{N}}{\partial y}\mathbf{v}_y\right]\mathrm{d}y\mathrm{d}x+\int_0^{\Delta x}\int_0^{\Delta y}\frac{\partial \mathbf{N}^T}{\partial y}\eta\left(\frac{\partial \mathbf{N}}{\partial y}\mathbf{v}_x+\frac{\partial \mathbf{N}}{\partial x}\mathbf{v}_y\right)\mathrm{d}x\mathrm{d}y-\int_0^{\Delta x}\int_0^{\Delta y}\frac{\partial \mathbf{N}^T}{\partial x}\mathbf{N}_p\mathbf{p}\,\mathrm{d}x\mathrm{d}y=\int_0^{\Delta x}\int_0^{\Delta y}\mathbf{N}^T F_x\mathrm{d}x\mathrm{d}y$$

$$\int_0^{\Delta y}\int_0^{\Delta x}\frac{\partial \mathbf{N}^T}{\partial y}\left[-\frac{2}{3}\eta\frac{\partial \mathbf{N}}{\partial x}\mathbf{v}_x+\frac{4}{3}\eta\frac{\partial \mathbf{N}}{\partial y}\mathbf{v}_y\right]\mathrm{d}x\mathrm{d}y+\int_0^{\Delta x}\int_0^{\Delta y}\frac{\partial \mathbf{N}^T}{\partial x}\eta\left(\frac{\partial \mathbf{N}}{\partial y}\mathbf{v}_x+\frac{\partial \mathbf{N}}{\partial x}\mathbf{v}_y\right)\mathrm{d}y\mathrm{d}x-\int_0^{\Delta x}\int_0^{\Delta y}\frac{\partial \mathbf{N}^T}{\partial y}\mathbf{N}_p\mathbf{p}\,\mathrm{d}x\mathrm{d}y=\int_0^{\Delta x}\int_0^{\Delta y}\mathbf{N}^T F_y\mathrm{d}x\mathrm{d}y$$

- Take nodal values out of integration

$$\int_0^{\Delta x}\int_0^{\Delta y}\frac{\partial \mathbf{N}^T}{\partial x}\frac{4}{3}\eta\frac{\partial \mathbf{N}}{\partial x}\mathrm{d}y\mathrm{d}x\,\mathbf{v}_x-\int_0^{\Delta x}\int_0^{\Delta y}\frac{\partial \mathbf{N}^T}{\partial x}\frac{2}{3}\eta\frac{\partial \mathbf{N}}{\partial y}\mathrm{d}y\mathrm{d}x\,\mathbf{v}_y+\int_0^{\Delta y}\int_0^{\Delta x}\frac{\partial \mathbf{N}^T}{\partial y}\eta\frac{\partial \mathbf{N}}{\partial y}\mathrm{d}x\mathrm{d}y\,\mathbf{v}_x$$
$$+\int_0^{\Delta y}\int_0^{\Delta x}\frac{\partial \mathbf{N}^T}{\partial y}\eta\frac{\partial \mathbf{N}}{\partial x}\mathrm{d}x\mathrm{d}y\,\mathbf{v}_y-\int_0^{\Delta x}\int_0^{\Delta y}\frac{\partial \mathbf{N}^T}{\partial x}\mathbf{N}_p\mathrm{d}x\mathrm{d}y\,\mathbf{p}=\int_0^{\Delta x}\int_0^{\Delta y}\mathbf{N}^T F_x\mathrm{d}x\mathrm{d}y$$

$$-\int_0^{\Delta y}\int_0^{\Delta x}\frac{\partial \mathbf{N}^T}{\partial y}\frac{2}{3}\eta\frac{\partial \mathbf{N}}{\partial x}\mathrm{d}x\mathrm{d}y\,\mathbf{v}_x+\int_0^{\Delta y}\int_0^{\Delta x}\frac{\partial \mathbf{N}^T}{\partial y}\frac{4}{3}\eta\frac{\partial \mathbf{N}}{\partial y}\mathrm{d}x\mathrm{d}y\,\mathbf{v}_y+\int_0^{\Delta x}\int_0^{\Delta y}\frac{\partial \mathbf{N}^T}{\partial x}\eta\frac{\partial \mathbf{N}}{\partial y}\mathrm{d}y\mathrm{d}x\,\mathbf{v}_x$$
$$+\int_0^{\Delta x}\int_0^{\Delta y}\frac{\partial \mathbf{N}^T}{\partial x}\eta\frac{\partial \mathbf{N}}{\partial x}\mathrm{d}y\mathrm{d}x\,\mathbf{v}_y-\int_0^{\Delta x}\int_0^{\Delta y}\frac{\partial \mathbf{N}^T}{\partial y}\mathbf{N}_p\mathrm{d}x\mathrm{d}y\,\mathbf{p}=\int_0^{\Delta x}\int_0^{\Delta y}\mathbf{N}^T F_y\mathrm{d}x\mathrm{d}y$$

Force balance equation: Some reorganization

- Writing everything as one equation in vector-matrix form

$$\int_0^{\Delta x}\int_0^{\Delta y} \left[\begin{array}{c|c} \frac{\partial \mathbf{N}^T}{\partial x}\frac{4}{3}\eta\frac{\partial \mathbf{N}}{\partial x} + \frac{\partial \mathbf{N}^T}{\partial y}\eta\frac{\partial \mathbf{N}}{\partial y} & \frac{\partial \mathbf{N}^T}{\partial x}\left(-\frac{2}{3}\eta\right)\frac{\partial \mathbf{N}}{\partial y} + \frac{\partial \mathbf{N}^T}{\partial y}\eta\frac{\partial \mathbf{N}}{\partial x} \\ \hline \frac{\partial \mathbf{N}^T}{\partial y}\left(-\frac{2}{3}\eta\right)\frac{\partial \mathbf{N}}{\partial x} + \frac{\partial \mathbf{N}^T}{\partial x}\eta\frac{\partial \mathbf{N}}{\partial y} & \frac{\partial \mathbf{N}^T}{\partial y}\frac{4}{3}\eta\frac{\partial \mathbf{N}}{\partial y} + \frac{\partial \mathbf{N}^T}{\partial x}\eta\frac{\partial \mathbf{N}}{\partial x} \end{array} \right] \mathrm{d}x\mathrm{d}y \begin{Bmatrix} v_{x,1} \\ \vdots \\ v_{x,9} \\ v_{y,1} \\ \vdots \\ v_{y,9} \end{Bmatrix} - \int_0^{\Delta x}\int_0^{\Delta y} \begin{Bmatrix} \frac{\partial \mathbf{N}^T}{\partial x} \\ \frac{\partial \mathbf{N}^T}{\partial y} \end{Bmatrix} \mathbf{N}_p \mathrm{d}x\mathrm{d}y \begin{Bmatrix} p_1 \\ \vdots \\ p_4 \end{Bmatrix} = \dots$$

- Reorganization

$$\int_0^{\Delta x}\int_0^{\Delta y} \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_1}{\partial y} \\ 0 & \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} \\ \vdots & \vdots & \vdots \\ \frac{\partial N_9}{\partial x} & 0 & \frac{\partial N_9}{\partial y} \\ 0 & \frac{\partial N_9}{\partial y} & \frac{\partial N_9}{\partial x} \end{bmatrix} \begin{bmatrix} \frac{4}{3}\eta & -\frac{2}{3}\eta & 0 \\ -\frac{2}{3}\eta & \frac{4}{3}\eta & 0 \\ 0 & 0 & \eta \end{bmatrix} \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \cdots & \frac{\partial N_9}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & \cdots & 0 & \frac{\partial N_9}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \cdots & \frac{\partial N_9}{\partial y} & \frac{\partial N_9}{\partial x} \end{bmatrix} dxdy \begin{Bmatrix} v_{x,1} \\ v_{y,1} \\ \vdots \\ v_{x,9} \\ v_{y,9} \end{Bmatrix} - \int_0^{\Delta x}\int_0^{\Delta y} \begin{Bmatrix} \frac{\partial N_1}{\partial x} \\ \frac{\partial N_1}{\partial y} \\ \vdots \\ \frac{\partial N_9}{\partial x} \\ \frac{\partial N_9}{\partial y} \end{Bmatrix} \mathbf{N}_p dxdy \begin{Bmatrix} p_1 \\ \vdots \\ p_4 \end{Bmatrix} =$$

Force balance equation: The final equation

$$\underbrace{\int_0^{\Delta x}\int_0^{\Delta y} \underbrace{\begin{bmatrix} \partial_x N_1 & 0 & \partial_y N_1 \\ 0 & \partial_y N_1 & \partial_x N_1 \\ \vdots & \vdots & \vdots \\ \partial_x N_9 & 0 & \partial_y N_9 \\ 0 & \partial_y N_9 & \partial_x N_9 \end{bmatrix}}_{\mathbf{B}^T} \underbrace{\begin{bmatrix} \frac{4}{3}\eta & -\frac{2}{3}\eta & 0 \\ -\frac{2}{3}\eta & \frac{4}{3}\eta & 0 \\ 0 & 0 & \eta \end{bmatrix}}_{\mathbf{D}} \underbrace{\begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \cdots & \frac{\partial N_9}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & \cdots & 0 & \frac{\partial N_9}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \cdots & \frac{\partial N_9}{\partial y} & \frac{\partial N_9}{\partial x} \end{bmatrix}}_{\mathbf{B}} dxdy}_{\mathbf{K}} \underbrace{\begin{Bmatrix} v_{x,1} \\ v_{y,1} \\ \vdots \\ v_{x,9} \\ v_{y,9} \end{Bmatrix}}_{\mathbf{v}}$$

$$\underbrace{-\int_0^{\Delta x}\int_0^{\Delta y} \begin{Bmatrix} \frac{\partial N_1}{\partial x} \\ \frac{\partial N_1}{\partial y} \\ \vdots \\ \frac{\partial N_9}{\partial x} \\ \frac{\partial N_9}{\partial y} \end{Bmatrix} \mathbf{N}_p dxdy}_{\mathbf{G}} \underbrace{\begin{Bmatrix} p_1 \\ \vdots \\ p_4 \end{Bmatrix}}_{\mathbf{p}} = \underbrace{\int_0^{\Delta x}\int_0^{\Delta y} \begin{Bmatrix} N_1 F_x \\ N_1 F_y \\ \vdots \\ N_9 F_x \\ N_9 F_y \end{Bmatrix} dxdy}_{\mathbf{F}}$$

$$\mathbf{Kv} - \mathbf{Gp} = \mathbf{F}$$

Apply numerical integration using 9 integration points

Incompressibility equation: Steps ① – ④

- Original equation: $$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$
- Test functions (use pressure functions): $$\mathbf{N}_p = \begin{bmatrix} N_1^p & \dots & N_4^p \end{bmatrix}$$

① Multiplication with test functions: $$\mathbf{N}_p^T \frac{\partial v_x}{\partial x} + \mathbf{N}_p^T \frac{\partial v_y}{\partial y} = 0$$

② Spatial integration: $$\int_0^{\Delta x}\int_0^{\Delta y} \mathbf{N}_p^T \frac{\partial v_x}{\partial x} \mathrm{d}x\mathrm{d}y + \int_0^{\Delta x}\int_0^{\Delta y} \mathbf{N}_p^T \frac{\partial v_y}{\partial y} \mathrm{d}x\mathrm{d}y = 0$$

③ NO integration by parts!!!

④ Apply FE-approximation: $$\int_0^{\Delta x}\int_0^{\Delta y} \mathbf{N}_p^T \frac{\partial \mathbf{N}}{\partial x} \mathrm{d}x\mathrm{d}y \mathbf{v}_x + \int_0^{\Delta x}\int_0^{\Delta y} \mathbf{N}_p^T \frac{\partial \mathbf{N}}{\partial y} \mathrm{d}x\mathrm{d}y \mathbf{v}_y = 0$$

Incompressibility equation: Some reorganization

- Writing everything as one equation in vector-matrix form

$$\int_0^{\Delta x}\int_0^{\Delta y} \mathbf{N}_p^T \begin{bmatrix} \frac{\partial \mathbf{N}}{\partial x} & \frac{\partial \mathbf{N}}{\partial y} \end{bmatrix} \mathrm{d}x\mathrm{d}y \begin{Bmatrix} v_{x,1} \\ \vdots \\ v_{x,9} \\ v_{y,1} \\ \vdots \\ v_{y,9} \end{Bmatrix} = 0$$

- Reorganization

Apply numerical integration using 9 integration points

$$\int_0^{\Delta x}\int_0^{\Delta y} \mathbf{N}_p^T \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_1}{\partial y} & \cdots & \frac{\partial N_9}{\partial x} & \frac{\partial N_9}{\partial y} \end{bmatrix} \mathrm{d}x\mathrm{d}y \begin{Bmatrix} v_{x,1} \\ v_{y,1} \\ \vdots \\ v_{x,9} \\ v_{y,9} \end{Bmatrix} = 0$$

$$\mathbf{G}^T \mathbf{v} = 0$$

Final set of equations

- Force balance: $$\mathbf{K}\mathbf{v} - \mathbf{G}\mathbf{p} = \mathbf{F}$$
- Incompressibility: $$\mathbf{G}^T \mathbf{v} = \mathbf{0}$$
- Everything together: $$\begin{bmatrix} \mathbf{K} & -\mathbf{G} \\ \mathbf{G}^T & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{v} \\ \mathbf{p} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F} \\ \mathbf{0} \end{Bmatrix}$$
- Voilà: $$\hat{\mathbf{K}}\mathbf{a} = \hat{\mathbf{F}}$$

What's new?

- New physical parameters (viscosity instead of elastic parameters)
- Numerical grid and indexing is different because of bi-quadratic shape functions (9 nodes per element).
  So, you need to change
    - `EL_N`
    - `EL_DOF`

  You need to newly introduce
    - `EL_P`, which provides a relationship between element numbers and pressure-node numbers
- New set of shape functions (4 bi-linear for pressure stay the same, 9 bi-quadratic for velocity are new)
- New set of integration points (9 instead of 4)
- Calculate $\mathbf{G}$ inside the loop over integration points in a similar way as $\mathbf{K}$. For this you need the shape function derivatives in a vector format (instead of matrix $\mathbf{B}$).
- Put $\mathbf{K}$, $\mathbf{G}$, and $\mathbf{F}$ together to form the super-big $\mathbf{K}$ and super-big $\mathbf{F}$.
- Your solution is now a velocity. So, introduce a time increment Δt and update your coordinates as $\mathbf{GCOORD} = \mathbf{GCOORD} + \Delta t * \mathbf{velocity}$. Then you can write a time-loop around your code and run it for several time steps.

Problem to be solved

- Homogeneous medium
- Boundary conditions
  - Left and bottom: Free slip (i.e., no boundary-perpendicular velocity, boundary-parallel velocity left free)
  - Right and top: Free surface (both velocities left free)
- Calculate several time steps