Earth’s core and the geodynamo

Exercises for Lecture 8: Self-exciting dynamos

Given out on 12th April. To be handed in on 26th April.

1. Analysis of a disk dynamo.

The equations governing the electrodynamics of the self-exciting disk dynamo described in the lectures are,

\[ \mathcal{E} = \frac{\Omega}{2\pi} \Phi \text{ where } \Phi = \int_S B_z \, dS \quad \text{(Induced EMF)} \]

\[ \Phi = M I \quad \text{(Mutual induction between wire and disk)} \]

\[ \mathcal{E} = L \frac{dI}{dt} + RI \quad \text{(Changes in current produced by induction and resistance effects)} \]

where \( \mathcal{E} \) is the EMF induced by the motion of the disk (area \( S \), rotating with angular velocity \( \Omega \)) through a magnetic field \( \mathbf{B} \) which has component \( B_z \) perpendicular to the disk. \( \Phi \) is the magnetic flux linked through the disk, \( M \) is the mutual inductance between the wire and the disk, \( I \) is the current in the wire, \( R \) is the resistance in the circuit when current \( I \) flows and \( L \) is the self-inductance of the circuit.

(i) By eliminating \( \Phi \) and \( \mathcal{E} \) from these equations show that the equation controlling the system is,

\[ L \frac{dI}{dt} = \left( \frac{\Omega M}{2\pi} - R \right) I. \]

(ii) Find a solution to this equation for \( I(t) \) by separating the variables and integrating.

(iii) What is the condition that must be satisfied in order for \( I(t) \) to grow exponentially?

(iv) In a real disk dynamo will the current continue to grow indefinitely? Give reasons for your answer, assuming a steady external torque \( G \) drives the motion of the disk.

(remaining text continues on the next page)
\[ \dot{\varepsilon} = L \frac{dI}{dt} + RI \]

\[ \varepsilon = \frac{\Omega M}{2\pi} \]

\[ \phi = MI + \int_S B_y \, ds. \]

\[ \frac{Q M I}{2\pi} = L \frac{dI}{dt} + RI \]

\[ L \frac{dI}{dt} = \left( \frac{Q M}{2\pi} - R \right) I \]

\[ L \frac{1}{I} \frac{dI}{dt} = \frac{Q M}{2\pi} - R. \]

\[ \frac{\mu_0}{I_0} \frac{I}{I} = \frac{1}{L} \left( \frac{Q M}{2\pi} - R \right) t \]

\[ I(t) = I_0 \exp \left( \frac{1}{L} \left( \frac{Q M}{2\pi} - R \right) t \right) \]

ii) In order for the dynamics to grow the growth rate \( L \left( \frac{Q M}{2\pi} - R \right) \) must be \( > 0 \)

\[ R < \frac{Q M}{2\pi}. \]
iii) If we assume that the torque to rotate the disk is constant $\Rightarrow$ the input mechanical power is given by

$$P = G\Omega.$$

Since the disk has a finite resistance the current will produce Ohmic dissipation:

$$P = RI^2$$

When $RI^2 = G\Omega$ the current can't grow any more.

or when the torque from the Lorentz forces counter balance $G$

$$\tau = \int_0^a r \times (I \times B) \, dr = \frac{1}{2} IBa^2 \hat{\theta} = \frac{M \omega^2}{2\pi} \hat{\theta}$$

(remember that $B \pi a^2 = M \omega$)

The current stop to grow $\Rightarrow$ $\frac{dI}{dt} = 0$. 
from the equation for the current:

\[ L \frac{dI}{dt} + RI = \epsilon \quad \epsilon = \frac{2\pi M I}{2\pi} \]

\[ \Rightarrow R \frac{dI}{dt} = \frac{\Omega M_{\text{max}} I_{\text{max}}}{2\pi} \]

the growth of \( I \) and therefore of \( B \) and \( M \) stops when:

\[ M_{\text{max}} = \frac{R \Omega I_{\text{max}}}{\Omega} \]

At that stage the Lorentz torque is given by:

\[ \tau = M \frac{I^2}{2\pi} = \frac{R I^2}{\Omega} = G \]

which is strictly equivalent to:

\[ G \Omega = RI^2 \]
2. Analysis of $\alpha \omega$ and $\alpha^2$ dynamos

In Cartesian co-ordinates, consider a system in which the magnetic and velocity fields are independent of the $y$ direction, so can be expressed in the form $\mathbf{B} = (-\partial A/\partial z, B, \partial A/\partial x)$ and $\mathbf{u} = (-\partial \psi/\partial z, u_y, \partial \psi/\partial x)$ where $A$ and $\psi$ are scalar potential functions for the 2D magnetic field and flow respectively. From the magnetic induction equation, the following equations can be obtained for the evolution of $B$ and $A$,

$$\frac{\partial B}{\partial t} + \left( \frac{\partial \psi}{\partial x} \frac{\partial B}{\partial z} - \frac{\partial \psi}{\partial z} \frac{\partial B}{\partial x} \right) = \left( \frac{\partial A}{\partial x} \frac{\partial u_y}{\partial z} - \frac{\partial A}{\partial z} \frac{\partial u_y}{\partial x} \right) - \nabla \cdot (\alpha \nabla A) + \eta \nabla^2 B$$

$$\frac{\partial A}{\partial t} + \left( \frac{\partial \psi}{\partial x} \frac{\partial A}{\partial z} - \frac{\partial \psi}{\partial z} \frac{\partial A}{\partial x} \right) = \alpha B + \eta \nabla^2 A$$

(a) Set $\psi=0$, let $\alpha$ be a constant, $u_y = U'z$ (a constant shear) and ignore the $\alpha$ term in the equation for the evolution of $B$ (i.e. an $\alpha \omega$ model).

(i) By substituting wave solutions of the form $(A, B) = (\hat{A}, \hat{B}) e^{(pt - ik_x x + k_z z)}$ into the simplified equations, demonstrate that for wave solutions to exist the following condition must be satisfied,

$$(p + \eta k^2)^2 = -ik_x \alpha U' \text{ where } k^2 = k_x^2 + k_z^2$$

(ii) If this $\alpha \omega$ dynamo wave is confined to a plane layer of height $d$ then $k_z = \pi/d$ is the wavenumber in the $z$ direction associated with the critical growing mode that satisfies the necessary boundary conditions. In this case, derive an equation which defines the critical value of $k_x$ for which dynamo action will occur (i.e. when $\mathcal{R}e[p] = 0$)

(b) In a second scenario, we have $\psi = u_y = 0$ and $\alpha$ is again a constant, but the alpha terms remains in both governing equations. This is an $\alpha^2$ dynamo.

(i) Substituting the same wave solutions as above into the governing equations for the $\alpha^2$ dynamo, show that solutions must now satisfy the relation,

$$p = \pm \alpha k - \eta k^2$$

(ii) What is the condition that must be satisfied for this dynamo to exhibit exponential growth? Is this dynamo oscillatory or steadily growing?
\[ \frac{\partial B}{\partial t} = \frac{\partial A}{\partial x} U^t + \eta \nabla^2 B \]

\[ \frac{\partial A}{\partial t} = \alpha B + \eta \nabla^2 A \]

\[ \frac{\nabla B}{\nabla} = -i k_x \nabla A - \eta (k_x^2 + k_y^2) B \]

\[ \frac{\nabla A}{\nabla} = \alpha B - \eta (k_x^2 + k_y^2) A \]

\[ B = \left[ \frac{p + \eta k^2}{\alpha} \right] A \]

\[ \left[ \frac{p + \eta k^2}{\alpha} \right] A = -i k_x \nabla A - \eta k^2 \left[ \frac{p + \eta k^2}{\alpha} \right] A \]

\[ \left[ p + \eta k^2 \right]^2 = -i \alpha k_x U^t \]
i) Let's fix \( k_3 = \frac{\pi}{d} \)

\[
P + \eta k^2 = \pm i \frac{1}{\sqrt{2}} (1+i) \sqrt{\alpha k_x} U^1
\]

\[
= \pm \frac{1}{\sqrt{2}} \left[ i - 1 \right] \sqrt{\alpha k_x} U^1
\]

\[
P = \pm \frac{1}{\sqrt{2}} \sqrt{\alpha k_x} U^1 - \eta k^2 \pm \frac{1}{\sqrt{2}} \sqrt{\alpha k_x} U^1
\]

\[
\text{Re} \left[ \hat{p} \right] = 0 \Rightarrow \frac{1}{\sqrt{2}} \sqrt{\alpha k_x} U^1 = \eta \left[ k_x + \frac{\eta^2}{d^2} \right]
\]

b) i) \( \hat{D}_B = - \nabla \alpha \nabla A + \eta \nabla^2 B \).

\[
\partial_t A = \alpha B + \eta^2 \nabla^2 A
\]

\[
\nabla A = \begin{pmatrix} jk_x \hat{A} & 0 & i k_3 \hat{A} \end{pmatrix} \exp \left( pt + i \left( k_x x + k_3 z \right) \right)
\]

\[
\nabla \alpha \nabla A = - \alpha \left( k_x^2 + k_3^2 \right) \hat{A} \exp \left( pt + i k_x x + k_3 z \right)
\]
\( \hat{p} \hat{B} = \alpha k^2 \hat{A} + \eta k^2 \hat{B} \)

\( \hat{p} \hat{A} = \alpha \hat{B} + \eta k^2 \hat{A} \)

\( \hat{B} = \alpha^{-1} \left[ p + \eta k^2 \right] \hat{A} \)

\( \alpha^{-1} \left[ p + \eta k^2 \right]^2 = \alpha k^2 \)

\[ p + \eta k^2 = \pm \alpha k. \]

\( \text{i) the dynamo will grow if } \operatorname{Re}[p] > 0 \)

\( \text{it is oscillatory if } \operatorname{Im}[p] \neq 0 \) - since \( \alpha, \kappa \text{ and } \eta \text{ are real } \operatorname{Im}[p] = 0 \).

\[ p > 0 \Rightarrow \pm \alpha k - \eta k^2 > 0 \]

for \( k > 0 \Rightarrow \alpha > \eta k \)

since \( \eta > 0 \)

\( \alpha > \eta k \)

\Rightarrow \text{lower bound on the length scale.} \)