Onset of plane layer rotating convection.

We consider the onset of Rayleigh-Bénard convection in a rotating plane layer between the planes \( z = 0 \) and \( z = L \). Gravity is pointing in the negative \( z \)-direction. The rotation rate is \( \Omega \) and the top and bottom plate are kept at temperatures \( T_0 \) and \( T_1 \), respectively.

The (non-dimensional) equations governing perturbations on the conductive equilibrium state are given by

\[
\nabla \cdot \mathbf{u} = 0,
\]

\[
\frac{\partial \mathbf{u}}{\partial t} + 2E^{-1} \mathbf{\hat{z}} \times \mathbf{u} = -\nabla p + \nabla^2 \mathbf{u} + RaPr^{-1}T \mathbf{\hat{z}},
\]

\[
\frac{\partial T}{\partial t} - u_z = Pr^{-1}\nabla^2 T.
\]

where the non-dimensional parameters are defined as

\[
E = \frac{\nu}{\Omega L^2}, Ra = \frac{\alpha g \Delta T L^3}{\kappa \nu}, Pr = \frac{\nu}{\kappa},
\]

and where \( g, \alpha, \kappa, \nu \) denote gravity, thermal expansivity, thermal diffusivity and kinematic viscosity, respectively. Finally \( \Delta T = (T_1 - T_0) \) is the temperature contrast maintained across the layer.

(i) Transform the equations (1)-(3) above in a set of equations in which \( u_z, \xi = \mathbf{\hat{z}} \cdot \nabla \times \mathbf{u} \) and \( T \) are the unknowns. To this end, apply the curl operator once and twice to the momentum equation (2), and consider the \( z \)-component of each of the resulting expressions.

(ii) Now consider perturbations that satisfy impermeable and stress-free velocity boundary conditions and isothermal temperature boundary conditions; these take the form (with \( l = n\pi \) and \( n \) being integer):

\[
u = \tilde{u}_z \sin(lz) \exp(\sigma t + i(k_x x + k_y y)),
\]

\[
\xi = \tilde{\xi} \cos(lz) \exp(\sigma t + i(k_x x + k_y y)),
\]

\[
T = \tilde{T} \sin(lz) \exp(\sigma t + i(k_x x + k_y y))
\]

The Rayleigh number \( Ra_c(k_x, k_y, l) \) that characterizes the growth of a perturbation is formally defined as the value of \( Ra \) for which the growth rate, \( \text{Re}[\sigma] \), is zero. Now assume that \( \text{Im}[\sigma] = 0 \)
as well. By substituting (5)-(7) into your result of part (i), show that the Rayleigh number for the onset of steady convection for a given \( k_x, k_y, l \) is defined by

\[
Ra_c(k_x, k_y, l) = \frac{(2l/E)^2 + (k^2 + l^2)^3}{k^2},
\]

(8)

where \( k^2 = k_x^2 + k_y^2 \).

(iii) Holding \( l \) fixed, differentiate (8) with respect to \( k^2 \) to find a condition for the critical horizontal wave number \( k_C \) that minimizes \( Ra_c \). Substitute your result back in (8).

(iv) Assume that the critical \( k_C \) satisfies \( k_C \gg 1 \), and find an expression for the scaling of \( k_C \) of the form \( k_C = aE^b \) where \( a \) is a constant and \( b \) a power defining the scaling behavior. Substitute your result back into your answer of part (iii) to find a scaling law for the critical Rayleigh number as a function of \( E \).
Taking the curl of N-S we obtain:

\[ \frac{\partial}{\partial t} \nabla \times u + 2 \varepsilon^{-1} \nabla \times (\hat{\mathbf{z}} \times u) = \nabla^2 (\nabla \times u) + \nabla \cdot \mathbf{P} \cdot \nabla (\hat{\mathbf{z}}) \]

\[ \nabla \times (\hat{\mathbf{z}} \times u) = \hat{\mathbf{z}} \nabla (\nabla \cdot u) - (\hat{\mathbf{z}} \cdot \nabla) u + (\nabla \cdot \hat{\mathbf{z}}) u \]

\[ \nabla \times (T \hat{\mathbf{z}}) = T \nabla \times \hat{\mathbf{z}} + \nabla T \times \hat{\mathbf{z}} \]

\[ \Rightarrow \hat{\mathbf{z}} \cdot \nabla \times (T \hat{\mathbf{z}}) = 0 \]

The z-component is written:

\[ \Delta_t \xi - 2 \varepsilon^{-1} \frac{\partial u}{\partial y} = \Delta^2 \xi \]

taking the \( \nabla \times \nabla \times \) of N-S:

\[ \Delta_t \left[ \nabla \times \nabla \times u \right] - 2 \varepsilon^{-1} \nabla \times (\hat{\mathbf{z}} \cdot \nabla u) = \nabla^2 (\nabla \times \nabla \times u) + \nabla \cdot \mathbf{P} \cdot \nabla (\hat{\mathbf{z}}) \]

\[ \nabla \times \nabla \times u = -\nabla^2 u \]

\[ \nabla \times (\hat{\mathbf{z}} \cdot \nabla u) = \nabla \times (\partial_y u) = \partial_y \nabla \cdot u \]

\[ \nabla^2 (\nabla \times \nabla \times u) = -\nabla^4 u \]

\[ \nabla \times (\nabla T \times \hat{\mathbf{z}}) = -\hat{\mathbf{z}} \nabla \cdot \nabla T + (\hat{\mathbf{z}} \cdot \nabla) \nabla T - (\nabla \cdot \nabla) \hat{\mathbf{z}} \]
Finally:

\[ \nabla \times \mathbf{v}(x) \Rightarrow \]

\[- \frac{2}{\epsilon} \nabla^2 u_y - 2 \epsilon^{-1} \frac{\partial}{\partial y} \mathbf{v} \cdot \nabla u_y = \nabla^4 u_y + \text{Ra} \text{Pr}^{-1} \left( \frac{\partial}{\partial x} \nabla^2 T - \frac{\partial}{\partial y} \nabla^2 T \right). \]

The \( \hat{z} \) component is given by \( \hat{z} \cdot \nabla \times \mathbf{v}(x) \):

\[- \frac{2}{\epsilon} \nabla^2 u_y - 2 \epsilon^{-1} \frac{\partial}{\partial y} \frac{\partial}{\partial x} \nabla^2 u_y = \nabla^4 u_y + \text{Ra} \text{Pr}^{-1} \left( \frac{\partial}{\partial x} \nabla^2 T - \nabla^2 T \right) \]

The heat equation is already expressed with \( u_y \) and \( T \):

\[
\frac{\partial T}{\partial t} - u_y = \text{Pr}^{-1} \nabla^2 T
\]

(ii)

Before we insert the ansatz into we may want to rearrange the equations:

\[ \text{Pr} \rightarrow \left[ \frac{\partial}{\partial t} - \nabla^2 \right] u_y = 2 \epsilon^{-1} \frac{\partial u_y}{\partial y} \]

\[ \rightarrow \left[ \frac{\partial}{\partial t} - \text{Pr}^{-1} \nabla^2 \right] T = u_y. \]
Applying the operator \([\partial_x - \nabla^2] [\partial_t - P^{-1}_r \nabla^2]\) to (B) will allow us to derive an equation for \(U_y\) only.

\[
\frac{\partial}{\partial t} - \nabla^2 \left[ \frac{\partial}{\partial t} - P^{-1}_r \nabla^2 \right] \nabla^2 U_y + 4E^{-2} \left[ \frac{\partial}{\partial t} - P^{-1}_r \nabla^2 \right] \frac{\partial^2}{\partial y^2} U_y
\]

\[= Ra P^{-1}_r \nabla^2 [\frac{\partial}{\partial t} - \nabla^2] U_y.\]

Let's make use of the ansatz \(\delta - 7\) now.

\[
\frac{\partial}{\partial t} \rightarrow \delta
\]

\[
\nabla^2 \rightarrow -l^2 - k^2
\]

\[
\frac{\partial^2}{\partial y^2} \rightarrow -l^2
\]

It yields:

\[
\left[ \delta + l^2 + k^2 \right]^2 \left[ \delta + P^{-1}_r (l^2 + k^2) \right] (l^2 + k^2) + 4E^{-2} \left[ \delta + P^{-1}_r (l^2 + k^2) \right] l^2 = Ra P^{-1}_r k^2 \left[ \delta + l^2 + k^2 \right]
\]

At the onset of convection \(\delta = 0\), inserting this in (D) leads to
\[
\left[ l^2 + k^2 \right] P_r^{-1} + 4E^{-2} P_r^{-1} E^2 \left[ l^2 + k^2 \right] = \kappa_a P_r^{-1} k^2 \left[ l^2 + k^2 \right]
\]

\[
\Rightarrow \quad R_{ac} = \frac{4E^{-2} l^2 + \left[ l^2 + k^2 \right]^3}{k^2}
\]

Comment: In this calculation the mass conservation equation does not appear explicitly. However, it is implicitly used in the derivation of \( D \times N \)-s.

The faster you rotate the lower is \( E \). Hence the higher in \( R_{ac} \). The rotation has a stabilizing effect.

(iii) The horizontal wavelength \( k_c \) that minimizes \( R_{ac} \) corresponds to the most unstable mode. It is defined by:

\[
\frac{dR_{ac}}{dk} = 0
\]

Note that for \( k \neq 0 \),

\[
\frac{dR_{ac}}{dk} = \frac{dR_{ac}}{dk^2} \frac{2k}{dk} = 2k \frac{dR_{ac}}{dk^2}
\]

Hence the above condition can be conveniently recast as

\[
\frac{dR_{ac}}{dk^2} = 0
\]
\[ \frac{\partial^2 R_c}{\partial k_c^2} = -\frac{1}{k_c^4} \left[ 4 E^{-2} l^2 + \left( l^2 + k_c^2 \right)^3 \right] + 3 \frac{\left( l^2 + k_c^2 \right)^2}{k_c^2} = 0. \]

\[ \Rightarrow R_c = 3 \left( l^2 + k_c^2 \right)^2 \]

\[ \text{iv) let's assume } k_c \gg 1, \text{ the equation for } k_c \text{ can be simplified as:} \]

\[ 4 E^{-2} l^2 + k_c^6 = 3 k_c^6 \]

\[ k_c \sim \left( 2 l^2 \right)^{1/6} E^{-1/3} \]

Injecting \( k_c \) into \( R_c \) we finally obtain the critical \( R_c \).

\[ R_c \sim 3 k_c^4 = 3 \left( 2 l^2 \right)^{2/3} E^{-4/3} \]