Theoretical Geomagnetism

Lecture 3

Core Dynamics I: Rotating Convection in Spherical Geometry
3.0 Ingredients of core dynamics

- Rotation places constraints on motions.

- Thermal (and chemical buoyancy) produces convection.

- Spherical shell geometry.

- Dynamo generated magnetic field will also influence motions.
3.0 Ingredients of core dynamics

- Buoyancy-driven flows and how such convective motions can generate dynamo action.
- Core flows are also strongly affected by Coriolis forces and Lorentz forces. In cases where the Coriolis forces dominate over the Lorentz forces, fluid motions tend to become two-dimensional along the direction of the rotation axis.
- It is estimated that Coriolis and Lorentz forces tend to be comparable in planetary core settings based on theoretical arguments (e.g., Fearn 1998) and from extrapolations of planetary magnetic field observations (e.g., Stevenson 2003). However, numerical simulations have shown that nearly two-dimensional, axially-aligned motions persist even for strong magnetic fields, with Lorentz forces up to $\frac{10}{C^2}$ times stronger than Coriolis forces (Olson and Glatzmaier 1995, Busse 2002).
- Thus, it is reasonable to assume that non-magnetic rotating convection experiments provide insight into the dynamics of planetary core processes.

- The non-dimensional parameters that describe thermal convection in rapidly rotating spherical shells are defined in Table 1. Typical parameter values in present-day models are far from the estimated values for planetary cores. However, it may be possible to simulate the convection dynamics without having to replicate the planetary core parameter values. Many physical systems have well-defined, limiting 'asymptotic' behaviors at either large or small values of the control parameters. If models show that a system has such a limiting scaling behavior, then extrapolation to planetary parameter values may be justified.

- For example, the schematic in Figure 2 shows a typical function $f(x)$ that varies with control parameter $x$. For intermediate values of $x$, the function $f(x)$ varies strongly with $x$. Hexagonal symbols mark out a set of hypothetical experimental results for Figure 1. Schematic showing the two basic convective flow structures thought to exist in planetary cores: large-scale axisymmetric zonal flows and small-scale convection columns.
3.1 Equations governing core dynamics

3.2 Stabilizing influence of rotation

3.3 Buoyancy driven flows (without magnetic field)

3.4 Spherical shell geometry and boundary layers

3.5 Summary
3.1.1 The Navier-Stokes equation

\[
\rho \frac{\partial u}{\partial t} + \rho (u \cdot \nabla) u = -\nabla P + \rho \nu \nabla^2 u
\]

- Rate of change of flow momentum w.r.t. time
- Advection of flow momentum along with fluid
- Force due to pressure gradient
- Viscous force on fluid

(known together as flow inertia, i.e. the acceleration of a fluid parcel)
3.1.2 Adding rotation: The Coriolis Force

\[ \rho \frac{\partial u}{\partial t} + \rho (u \cdot \nabla) u + 2\rho (\Omega \times u) = -\nabla P + \rho \nu \nabla^2 u \]

- \( u \) is now the fluid velocity in the rotating reference frame.

Coriolis force due to rotating reference frame

Centrifugal acceleration is added to pressure gradient, so \( P \) is effective pressure in the rotating frame.
3.1.3 The Buoyancy force

\[
\frac{\rho_0}{\partial t} \frac{\partial u}{\partial t} + \rho_0 (u \cdot \nabla) u + 2\rho_0 (\Omega \times u) = -\nabla P - \rho_0 \alpha T g + \rho_0 \nabla^2 u
\]

- The effects of the buoyancy forces are most easily taken into account using the Boussinesq Approximation:
  - The fluid is assumed to have constant background density \( \rho_0 \) with a background temperature field \( T_0 \), and perturbations \( T \) evolving as:
    \[
    \frac{\partial T}{\partial t} + (u \cdot \nabla) T_0 = \kappa \nabla^2 T
    \]
  - Viscous and Ohmic heating effects neglected.
  - An additional buoyancy term and transport equation is needed to incorporate compositional convection (left away here for the sake of simplicity)
  - Only dynamic effect of \( T \) is through gravity \( g \) acting on density perturbations \( \rho \alpha T \) as described in the buoyancy force.

Temperature differences produce a change in density and so a buoyancy force.
3.1.4 The Lorentz force

\[ \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} + 2\rho(\Omega \times \mathbf{u}) = -\nabla P - \rho \alpha T \mathbf{g} + (\mathbf{J} \times \mathbf{B}) + \rho \nu \nabla^2 \mathbf{u} \]

Lorentz force: Force due to magnetic field acting on electric currents in fluid.

\[ \mathbf{J} \times \mathbf{B} = \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} = -\nabla \left( \frac{\mathbf{B}^2}{2\mu_0} \right) + \frac{1}{\mu_0} (\mathbf{B} \cdot \nabla)\mathbf{B} \]

• Lorentz force can be written as a sum of terms accounting for magnetic pressure and field line curvature effects.

• The evolution of the magnetic field is as before determined by the magnetic induction equation,

\[ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B} \]
3.1.5 Equations governing core dynamics

\[
\begin{align*}
\rho \frac{\partial u}{\partial t} + \rho (u \cdot \nabla) u + 2 \rho (\Omega \times u) &= -\nabla P - \rho \alpha T g + \nabla \times (J \times B) + \rho \nu \nabla^2 u \\
\frac{\partial T}{\partial t} + (u \cdot \nabla) T_0 &= \kappa \nabla^2 T \\
\frac{\partial B}{\partial t} &= \nabla \times (u \times B) + \eta \nabla^2 B \\
\nabla \cdot u &= 0 \\
\nabla \cdot B &= 0
\end{align*}
\]

Terms representing feedback and coupling between the velocity, magnetic and temperature fields

+ Boundary Flow, Thermal and Magnetic BC on ICB & CMB.

• These are the equations of convection-driven, rotating magnetohydrodynamics under the Boussinesq approx.

• To get some feeling for the flow patterns and dynamics possible for such a system, we will consider some simple examples.....
Lecture 3: Core Dynamics I

3.1 Equations governing core dynamics

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3.5 Summary
3.2.1 Geostrophic Balance

- For **slow motions** (so that inertial terms are negligible), and also **neglecting buoyancy** driving, **viscous** effects and the **Lorentz** force, then we can isolate the influence of rotation on fluid motions:

\[ 2(\Omega \times u) = -\frac{\nabla P}{\rho} \]

**Geostrophic Balance between the Coriolis force and the pressure gradient.**

- Flow is perpendicular to pressure gradient and involves circulations around regions of high and low pressure.
3.2.2 Taylor-Proudman Theorem

- Assuming the rotation vector is parallel to the z axis so \( \mathbf{\Omega} = \mathbf{\Omega} \hat{z} \),

\[
2(\mathbf{\Omega} \hat{z} \times \mathbf{u}) = -\frac{\nabla P}{\rho}
\]

- Taking the curl of both sides,

\[
2\Omega \frac{\partial \mathbf{u}}{\partial z} = -\frac{1}{\rho^2} (\nabla P \times \nabla \rho)
\]

- If density gradients are negligible so that \( \rho = \rho_0 \),

\[
\frac{\partial \mathbf{u}}{\partial z} = 0
\]

Taylor-Proudman theorem

i.e. In rapidly rotating fluids, flow patterns are INvariant // rotation axis. (or flow varies only in 2D, in planes perpendicular to rotation axis)
3.2.2 Taylor-Proudman Theorem

Non-rotating case

\[ \frac{\partial \mathbf{u}}{\partial z} \neq 0 \]

Rapidly-rotating case

\[ \Omega \]

\[ \frac{\partial \mathbf{u}}{\partial z} = 0 \]

Taylor-Proudman Theorem

(From Aurnou, Les Houches Lectures, 2007)
3.2.3 Geostrophic motions

• For incompressible fluids, in order for \( \partial u / \partial z = 0 \) then

\[
\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0
\]

• This requires z-independent flow patterns motions to follow contours of constant depth.

\[\text{Geostrophic Contours}\]

e.g. Flow patterns move only zonally at fixed latitude in a spherical shell without changing height \( H \).
3.2.4 Quasi-geostrophic motions

• Forcing by convection, inhomogeneous boundary conditions or magnetic effects will lead to departures from perfect geostrophy.

• When columnar disturbances are forced off geostrophic contours the resulting motions are known as quasi-geostrophic.

• To understand such motions it is instructive to begin with the Navier-Stokes equation for a rotating fluid in the absence of any forcing (e.g. buoyancy or Lorentz) and without viscous dissipation:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + 2(\Omega \times u) = -\frac{\nabla P}{\rho_0}$$

• Curling this we obtain an evolution eqn for the vorticity $\xi = \nabla \times u$:

$$\frac{\partial \xi}{\partial t} + (u \cdot \nabla)\xi - (\xi \cdot \nabla)u - 2\Omega \frac{\partial u}{\partial z} = 0$$

• This can be re-written in the form of a ‘frozen flux’ eqn for $\xi + 2\Omega$,

$$\frac{D(\xi + 2\Omega)}{Dt} = [(\xi + 2\Omega) \cdot \nabla]u$$

(1)
3.2.4 Quasi-geostrophic motions

• To progress we adopt a quasi-geostrophic scaling for \( \mathbf{u} \) whereby motions are 2D in the \((x,y)\) plane perpendicular to the rotation axis (i.e. geostrophic) but there is also a weaker flow parallel to the rotation axes, which can vary in this \( \hat{z} \) direction:

\[
\mathbf{u} = u(x, y) \hat{x} + v(x, y) \hat{y} + w(x, y, z) \hat{z} \quad \text{where} \quad w << u, v
\]

• The approximately 2D flow can therefore be represented in terms of a stream-function representation in terms of a single scalar:

\[
(u, v, 0) = \left( \frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}, 0 \right)
\]

• Substituting into (1), then to leading order the \( \hat{z} \) (axial) component of the vorticity equation is,

\[
\frac{D(\xi_z + 2\Omega)}{Dt} = (\xi_z + 2\Omega) \frac{\partial w}{\partial z}
\]
3.2.4 Quasi-geostrophic motions

- Integrating over the \( \hat{z} \) direction, this yields an expression depending on the axial flow at the boundaries,

\[
H \frac{D(\xi_z + 2\Omega)}{Dt} = (\xi_z + 2\Omega)[w(H) - w(0)]
\]

- Due to non-penetration at the boundaries,

\[
\frac{DH}{Dt} = [w(H) - w(0)]
\]

- So,

\[
H \frac{D(\xi_z + 2\Omega)}{Dt} - (\xi_z + 2\Omega) \frac{DH}{Dt} = 0
\]

- Or,

\[
\frac{D}{Dt} \left( \frac{\xi_z + 2\Omega}{H} \right) = 0 \tag{2}
\]

- i.e. in the absence of forcing or dissipation, potential vorticity is conserved moving with the fluid.
Since \( \eta \cdot \eta = 0 \) and \( \Phi \cdot \eta = 0 \) the variation of \( H \) correspond to material expelled from the column.
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3.4 Spherical shell geometry and boundary layers
3.5 Summary
3.3.1 Convection-driven flows: Motivation

• Is the Earth’s core convecting?
• How do we know/can we investigate this?
3.3.2 Convection in a rotating plane layer

The equations describing the physical behavior of this system are:

\[ \nabla \cdot \mathbf{u} = 0 \]

\[ \rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + 2\Omega \hat{z} \times \mathbf{u} \right) = -\nabla p + \rho \nu \nabla^2 \mathbf{u} + \rho \alpha T \mathbf{g} \hat{z} \]

\[ \frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \kappa \nabla^2 T \]

A purely conductive solution is:

\[ \mathbf{u} = 0, \ T = T_1 + (T_0 - T_1) \frac{\hat{z}}{L}, \ p = \rho \alpha g \left( T_1 z + (T_0 - T_1) \frac{z^2}{2L} \right) \]
3.3.2 Convection in a rotating plane layer

- The question is now whether this conductive equilibrium state is dynamically stable.
- Stability is related to the evolution of small (infinitesimal) perturbations of the equilibrium state. If they are exponentially amplified, then the equilibrium is said to be unstable.
- Thus, we consider perturbations of the form (with $\varepsilon \ll 1$)

\[
\begin{align*}
\mathbf{u} &= U + \varepsilon \mathbf{u}', \\
p &= P + \varepsilon p', \\
T &= \Theta + \varepsilon T'
\end{align*}
\]
3.3.2 Convection in a rotating plane layer

Neglecting terms in $\varepsilon^2$ w.r.t. $\varepsilon$ and omitting the ′, we obtain the following (linear) perturbation equations:

\[ \nabla \cdot \mathbf{u} = 0 \]

\[ \rho \left( \frac{\partial \mathbf{u}}{\partial t} + 2\Omega \hat{z} \times \mathbf{u} \right) = -\nabla p + \rho \nu \nabla^2 \mathbf{u} + \rho \alpha g T \hat{z} \]

\[ \frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla \Theta = \kappa \nabla^2 T \]

$\varepsilon^2$ $\mathbf{u} \cdot \nabla T$ is $O(\varepsilon^2)$. 
3.3.2 Convection in a rotating plane layer

- It’s customary to write these equations in non-dimensional form with

\[
E = \frac{\nu}{\Omega L^2}, \quad Ra = \frac{\alpha g (T_1 - T_0) L^3}{\nu \kappa}, \quad Pr = \frac{\nu}{\kappa}
\]

\[
\nabla \cdot \mathbf{u} = 0
\]

\[
\frac{\partial \mathbf{u}}{\partial t} + 2 E^{-1} \hat{z} \times \mathbf{u} = - \nabla p + \nabla^2 \mathbf{u} + Ra Pr^{-1} T \hat{z}
\]

\[
\frac{\partial T}{\partial t} - u_z = Pr^{-1} \nabla^2 T
\]
We use \((T_r - T_0)\) as a temperature scale.

\[ L \quad \text{Length} \quad \rho \quad \text{Time} \quad \eta \]

\[ u \cdot \nabla \Theta = u_3 \frac{\partial}{\partial z} \Theta \quad \text{Since } \Theta \text{ varies only in } z. \]

\[ = u_3 (T_r - T_0) \]

Using the typical temperature scale,

\[ u^{+} \hat{\nabla}^{+} = \frac{u}{L} \frac{1}{\Delta t} \nabla \left( \frac{\Theta(z)}{\Theta_T} \right) = u^{+} \frac{\partial}{\partial z} \left( \frac{T_r - T_0}{T_r - T_0} \right) \]

\[ \nabla = \frac{1}{L} \hat{\nabla}^{+} \quad u = u^{+} \Delta L \]
3.3.2 Convection in a rotating plane layer

- Solutions to this set of equations are found by assuming the following form for the perturbations:

\[ \{u, p, T\} = \{\tilde{u}(z), \tilde{p}(z), \tilde{T}(z)\} \exp(\sigma t + i(k_x x + k_y y)) \]

- Imposing of the (velocity) boundary conditions at the top and bottom wall will lead to a constraint:

\[ \sigma = f(k_x, k_y, E, Pr, Ra) \]

- Instability will occur if

\[ \sigma^* = \max_{k_x, k_y} \text{Re}[\sigma] > 0 \]
\[ x \leq E^{1/3} \leq 10^{-5} \]

\[ \lambda (m) = R \times 10^{-5} \times 3 \times 10^{3} \times 10^{-5} \times 10^{3} \]

30 m.
3.3.2 Convection in a rotating plane layer

- After a lengthy calculation, it is found for free-slip BCs that instability occurs if,

\[ Ra \gtrsim 8.3E^{-4/3} \]

and the wavelength \( k \) at the onset of instability scales as:

\[ k \sim E^{-1/3} \]

- What’s the temperature drop needed to have convection for Earth-like conditions?
3.3.2 Convection in a rotating plane layer

\[ Ra = 8 \cdot 10^7, E = 10^{-5}, Pr = 1 \]
3.3.2 Convection in a rotating plane layer

\[ Ra = 8 \cdot 10^7, E = 10^{-5}, Pr = 1 \]

\[ Ra = 1.5 \cdot 10^9, E = 10^{-5}, Pr = 1 \]
3.3.3 Rotating convection in spherical shells

• In a rotating spherical shell heat is again transported by **columnar convection**.
• Consists of a series of slowly drifting columns.
• Well understood with combination of theory, experiments & simulation.
3.3.3 Rotating convection in spherical shells

(N.B. Geometry in this example movie is not that in core, except maybe inside the tangent cylinder. Here we have a simple plane layer with cold top and hot bottom)

- Fundamental mechanism consists of pairs of cyclonic and anti-cyclonic cells which involve upwelling and downwellings respectively at the outer boundary. These transport heat from the hot ICB to the colder CMB.

- Mechanism involves viscosity breaking the Taylor-Proudman constraint allowing heat to be transported: viscosity enters the vorticity balance due to the small azimuthal and radial length scale (tall, thin columns) and also in the viscous (Ekman) boundary layer.
3.3.3 Rotating convection in spherical shells

Spiralling cyclonic and anti-cyclonic columnar convection cells transport heat in rapidly rotating spherical systems (Busse, 1970)
3.3.5 Non-linear convection

- We have now established that the critical Rayleigh number $R \alpha_c$ above which convection occurs scales as:

$$R \alpha_c \sim E^{-4/3}$$

- Applied to Earth's core conditions, this corresponds to an ICB/CMB (super-adiabatic) temperature contrast of the order of microkelvin.

- It is customary to quantify the vigor of convection in terms of the Nusselt number, which is the ratio between the total heat flux at the boundary and the conductive heat flux, e.g. for a plane layer:

$$Nu = \left\langle \frac{\Phi|_{z=\text{top}}}{\rho c_p \kappa \frac{\Delta T}{L}} \right\rangle = - \frac{L}{\Delta T} \left\langle \frac{\partial T}{\partial z} \right\rangle$$

- We now would like to establish scaling laws of the form:

$$Nu = f(Ra, Pr, E)$$
3.3.5 Non-linear convection

- More vigorous convection leads to enhanced mixing, and thus we expect the fluid to become more isothermal.

- Near the boundaries however, the velocity has to tend to zero, and therefore the (advective) transport of heat is inhibited. Within these boundary layers, heat transport is mainly conductive.

- In non-rotating convection, the boundary layer thickness is such that it remains marginally stable, i.e.: \( \delta \sim \left( \frac{Ra}{Ra_c} \right)^{-1/3} \)
What is the thickness of the boundary layer?

- Express the thickness $\delta$ as a function of the critical Rayleigh number $Ra_c$.

- Assuming that the same heat flux goes thru the top and bottom boundary layer and that the temperature is uniform in the interior, express the temperature gradient across the thermal boundary layers $\Delta T$ as a function of $T_1$ and $T_2$.

- Demonstrate that

\[
\frac{\delta}{H} \propto \left( \frac{Ra}{Ra_c} \right)^{-1/3}
\]
What is the thickness of the boundary layer?

Looking at the boundary layer as a plane layer of thickness $\delta$:

$$Ra_\delta = \frac{\alpha g \Delta T_\delta \delta^3}{\nu \kappa} = Ra_c$$

Hence,

$$\delta = \left[ Ra_c \frac{\nu \kappa}{\alpha g \Delta T_\delta} \right]^{1/3}$$

Since the temperature outside of the two boundary layers is uniform it yields:

$$\Delta T_\delta = \frac{\Delta T}{2}.$$
What is the heat flux through the boundary layers?

You know that all the heat coming from the bottom must travel vertically and leave throughs the top surface.

Express the Nusselt number as a function of $\delta, \Delta T, H$ and the thermal conductivity $k$ of the fluid.

- Demonstrate that

\[
Nu \propto \left( \frac{Ra}{Ra_c} \right)^{1/3}
\]
One particular case where Nu can be estimated

Looking at the boundary layer as a plane layer of thickness $\delta$:

$$Ra_\delta = \frac{\alpha g \Delta T_\delta \delta^3}{\nu \kappa} = Ra_c$$  \hspace{1cm} (144)

Hence,

$$\delta = \left[ Ra_c \frac{\nu \kappa}{\alpha g \Delta T_\delta} \right]^{1/3}$$  \hspace{1cm} (145)

Since the temperature outside of the two boundary layers is uniform it yields:

$$\Delta T_\delta = \frac{\Delta T}{2}.$$  \hspace{1cm} (146)

Recalling that $Ra = \frac{\alpha g \Delta T H^3}{\nu \kappa}$ we obtain:

$$\frac{\delta}{H} \propto \left( \frac{Ra}{Ra_c} \right)^{-1/3}$$  \hspace{1cm} (147)

We know that all the heat that has been advected upward must traverse the boundary layers at the top and bottom:

$$Q_{total} = Q_{BL}$$
One particular case where $\text{Nu}$ can be estimated

Recalling that $Ra = \frac{\alpha g \Delta T H^3}{\nu \kappa}$ we obtain:

$$\frac{\delta}{H} \propto \left( \frac{Ra}{Ra_c} \right)^{-1/3}$$

$$\dot{Q}_{BL} \sim k \frac{\Delta T/2}{\delta} = k \frac{\Delta T/2}{H} \left( \frac{Ra}{Ra_c} \right)^{1/3}$$

$$\dot{Q}_{conduction} \sim k \frac{\Delta T}{H}$$

$$Nu \propto \left( \frac{Ra}{Ra_c} \right)^{1/3}$$
3.3.5 Non-linear convection

Experiments in rotating fluids show the following behavior:
- Close to onset, the Nusselt number scales as $Nu \sim Ra^{3}E^{4}$. This implies that the Ekman boundary layer is marginally stable.
- For highly supercritical convection, the non-rotating scaling law is recovered: $Nu \sim Ra^{3}$. (King et al., JFM 2012)

(King et al., JFM 2012)
\[ Nu = \frac{Q_{\text{total}}}{Q_{\text{cond.}}}, \text{ same for both rotating and non rotating.} \]

\[ Nu = 25 \quad \text{Vs.} \quad Nu = 4 \rightarrow 4 \text{ times more efficient} \]
3.3.6 Thermal winds

- In rotating fluids, if large density variations are present (for example due to strong temperature anomalies) then flows or winds involving shear can result:

\[
\frac{\partial u}{\partial z} = -\frac{g\alpha}{2\Omega} (\nabla \times T\hat{r})
\]

- Considering the component of this in the east-west (azimuthal) direction using spherical polar co-ordinates, we can see how a large temperature gradient in the latitudinal direction can produce a shear flow in the axial direction.

\[
\frac{\partial u_\phi}{\partial z} = \frac{g\alpha}{2\Omega r} \frac{1}{\partial \theta} \frac{\partial T}{\partial \theta}
\]

- Such ‘thermal winds’ may exist in Earth’s core, for example inside the tangent cylinder above and below the inner core.
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Lecture 3: Core Dynamics I

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3.5 Summary: self-assessment questions

(1) What are the equations governing core dynamics and can you describe what each term in these equation means?

(2) What is geostrophy and can you derive the Taylor-Proudman theorem?

(3) How do quasi-geostrophic (Rossby) waves operate?

(4) What form does convection take in a rapidly-rotating spherical shell?

(5) What form of boundary layers are present in the core?

Next time: *Influence of magnetic fields on core motions* (Magnetic winds, Taylor’s constraint, Alfven waves)
References


