

Inertial modes in a rotating triaxial ellipsoid

S. Vantieghem¹

¹*Institut für Geophysik, ETH Zürich, Sonneggstrasse 5, CH8092 Zürich**

Abstract

In this work, we present an algorithm that enables computation of inertial modes and their corresponding frequencies in a rotating triaxial ellipsoid. The method consists of projecting the inertial mode equation onto finite-dimensional bases of polynomial vector fields. It is shown that this leads to a well-posed eigenvalue problem, and hence, that eigenmodes are of polynomial form. Furthermore, these results shed new light onto the question whether the eigenmodes form a complete basis, i.e. whether any arbitrary velocity field can be expanded in a sum of inertial modes. Finally, we prove that two intriguing integral properties of inertial modes in rotating spheres and spheroids also extend to triaxial ellipsoids.

* stijn.vantieghem@erdw.ethz.ch

I. INTRODUCTION

One of the most striking features of incompressible, rotating flows is their ability to sustain wave motion. The restoring force giving birth to these waves is the Coriolis force. Because this phenomenon only relies on inertia, eigenmodes of wave propagation in bounded geometries bear the name inertial modes. Numerous studies, both computational and experimental, have shown that these waves can be excited and maintained in different ways, e.g. by thermal and shear instabilities [1, 2], libration [3], precession [4–6], tides [7] or differential rotation [8, 9].

The Coriolis force being dominant in the force balance that governs the dynamics of numerous geo- and astrophysical bodies, it is widely believed that inertial modes are fundamental concepts for our understanding of many of these bodies' properties. For instance, inertial modes emanate at the onset of small Prandtl number convection in a rapidly rotating sphere subject to volumetric heating [1], as is the case in astrophysics. Furthermore, several authors have argued that the ancient lunar dynamo[10] and Io's magnetic signature [11] are generated by complex fluid motions that arise as a result of a so-called elliptic instability. This phenomenon can be understood as the resonant coupling between two inertial modes and a strain field due to the tidal deformation of the celestial body under consideration [12].

To verify and clarify these mechanisms, a number of studies have been devoted to the dynamic response of fluid-filled rotating triaxial ellipsoids to harmonic forcings, such as libration [13–15], precession [16] and tides [17]. However, neither laboratory experiments nor numerical simulations can reach the extremely low values of viscosity that characterizes the dynamics of planetary cores. The strength of viscous effects is usually quantified by means of the Ekman number, a non-dimensional estimate of the ratio between the viscous force and the Coriolis force. In planetary settings, the value of the Ekman number typically takes values in the range $10^{-15} - 10^{-10}$. For the laboratory experiments cited, this quantity has order-of-magnitude $10^{-6} - 10^{-4}$. Numerical studies even have to compromise further on the value of the Ekman number. To compensate for this discrepancy, the aforementioned studies have been carried out in enclosures whose ellipticity is much larger than that is encountered on a geo- and astrophysical scale. However, existing theories are built upon the assumption that the equatorial deformation is small such that the mathematical analysis can be pursued in terms of spheroidal inertial modes[17, 18]. To bridge the gap between theory

on the one hand, and simulations and experiments on the other hand, it would be convenient to deduce an analytical expression for inertial modes in triaxial ellipsoidal geometry. This is the context in which this work was conceived. Apart from this, the problem of inertial modes in a rotating triaxial ellipsoid is also an interesting mathematical question in its own right.

Pioneering work regarding the theory of inertial modes was established by Lord Kelvin [19], who derived an expression for inertial oscillations in a cylindrical geometry. Shortly afterwards, Poincaré [20] formulated the generic mathematical theory describing such oscillations. Bryan[21] on the other hand provided a general implicit expression for the inertial modes in a spheroidal geometry. Note that we use the terminology ‘spheroid’ to signify an ellipsoid whose two axis perpendicular to the rotation axis are equal. These theories were rediscovered independently by Bjerknes and co-workers [22], who termed them ‘elastoid-inertia’ oscillations. Later contributions have focused on the effect of the presence of a finite but small amount of viscosity. A major *tour de force* in this context is the expression for the corrective boundary layer in a spherical and a spheroidal geometry [23–25]. In the same era, Kudlick also proposed a procedure to calculate explicitly spheroidal inertial modes; this however involves finding the roots of a polynomial of high-degree, and thus requires numerical work. More recently, Kerswell [16] showed that spheroidal inertial modes could be constructed from finite-dimensional bases of polynomial vector fields. Zhang and co-workers on the other hand, were the first ones to obtain explicit expressions for all possible inertial modes that can exist in a spherical and spheroidal geometry [26, 27]. Furthermore, much attention has been devoted to non-smooth inertial wave solutions that can be excited in a spherical shell[28–30] or parallelepiped[31]; in the low-viscosity limit, these take the form of wave attractors.

This work builds upon and extends the existing theories of inertial modes in the following sense: (i) We outline a procedure to compute inviscid inertial modes in a triaxial ellipsoid. We apply this technique to compute, for the first time, a handful of eigenvalues and eigenmodes in triaxial geometry. In this sense, our contribution is somewhat similar to the results obtained by Kudlick[25] for inertial modes in a spheroid. It is also an extension of the approach of Kerswell[16] to triaxial ellipsoids. Furthermore, following Greenspan[32], an expression for the viscous corrections to the eigenvalue is at hand, once the inviscid problem has been solved. In this work, we will compute these corrections for one particular class of

modes. (ii) We shed new light on the still open question whether the inertial modes form a complete set, i.e. whether it is possible to expand an arbitrary, sufficiently smooth solenoidal vector field in a series of inertial modes. (iii) We prove that two integral properties of inertial modes, originally discovered by Zhang, Liao et al. [27, 33] for rotating spheres, extend to triaxial ellipsoids.

II. MATHEMATICAL FORMULATION

We consider an inviscid and incompressible fluid contained within a triaxial ellipsoid of semi-major axes a , b and c . In mathematical terms, it is described by:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \quad (1)$$

The notations x, y, z refer to the cartesian coordinates. In the following, we will denote the position vector as \mathbf{r} , i.e. $\mathbf{r} = (x, y, z)$. The ellipsoid is rotating at angular speed Ω_0 around the z -axis. We now consider the equations of mass and momentum conservation for small amplitude motion of an inviscid compressible fluid, expressed within a frame of reference rotating with the ellipsoid:

$$\nabla \cdot \mathbf{U} = 0, \quad (2)$$

$$\frac{\partial \mathbf{U}}{\partial \tau} + 2\hat{\mathbf{z}} \times \mathbf{U} + \nabla P = 0. \quad (3)$$

Here, τ denotes time with respect to the time scale Ω_0^{-1} , $\hat{\mathbf{z}}$ is the unit vector along the rotation axis, and \mathbf{U} and P are the velocity and pressure field. We have dropped the non-linear term in the momentum equation (3), as we assume small amplitude motion. Inviscid inertial modes are then solutions of (2)-(3) of the form $\mathbf{U} = \mathbf{u}(\mathbf{r}) \exp(i\lambda\tau)$, $P = p(\mathbf{r}) \exp(i\lambda\tau)$. The scalar λ is termed the *eigenvalue* or *eigenfrequency*, and the fields $\{\mathbf{u}, p\}$ *eigenmodes*. As such, the above equations can be reformulated as follows:

$$\nabla \cdot \mathbf{u} = 0, \quad (4)$$

$$2\hat{\mathbf{z}} \times \mathbf{u} + \nabla p = -i\lambda\mathbf{u}. \quad (5)$$

Finally, the system of equations (2)-(3) or (4)-(5) should be supplemented with the non-penetration condition at the boundary of the ellipsoidal volume:

$$\mathbf{U} \cdot \hat{\mathbf{n}} = \mathbf{u} \cdot \hat{\mathbf{n}} = 0, \quad (6)$$

where the vector $\hat{\mathbf{n}}$ denotes the unit outward normal to the triaxial ellipsoid:

Equation (5) can be recast in terms of the vorticity $\boldsymbol{\omega} \equiv \nabla \times \mathbf{u}$.

$$\nabla \times (2\hat{\mathbf{z}} \times \mathbf{u}) = -i\lambda\boldsymbol{\omega}. \quad (7)$$

We note that the non-penetration condition (6) does not impose any constraint on the vorticity field.

It can be shown that the eigenvalues λ are real and $-2 \leq \lambda \leq 2$ [32]. Furthermore, eigenmodes \mathbf{u} satisfy the following orthogonality relationship:

$$\iiint \mathbf{u}_i^\dagger \cdot \mathbf{u}_j \, dV = \delta_{ij}, \quad (8)$$

provided that $\lambda_i \neq \lambda_j$. The notations \dagger and δ_{ij} respectively refer to the complex conjugate and the Kronecker delta. In this expression, and in the remainder of this article, the integration domain for three-dimensional integrals is always the ellipsoidal volume bounded by (1).

III. AN ALGORITHM TO FIND INERTIAL MODES

In this section, we will develop a method to compute inviscid inertial modes in a triaxial ellipsoid. Most authors that have addressed this problem in other geometries, did so by recasting the eigenvalue problem (4)-(6) in terms of the pressure alone. This yields the famous Poincaré equation[20]:

$$\nabla^2 p - \left(\frac{\lambda}{2}\right)^2 (\hat{\mathbf{z}} \cdot \nabla)^2 p = 0, \quad (9)$$

supplied with the boundary condition:

$$(\hat{\mathbf{z}} \cdot \nabla p) (\hat{\mathbf{z}} \cdot \hat{\mathbf{n}}) - \left(\frac{\lambda}{2}\right)^2 \hat{\mathbf{n}} \cdot \nabla p - i\frac{\lambda}{2} (\hat{\mathbf{n}} \times \hat{\mathbf{z}}) \cdot \nabla p = 0. \quad (10)$$

For $-2 \leq \lambda \leq 2$, this equation is hyperbolic and is in general ill-posed; this implies that it has singular solutions that are not square-integrable [28, 34]. However, for specific geometries, such as cylinders, spheres and spheroids[21, 25], and annular ducts [35], the problem is well-posed and regular solutions for the pressure eigenmodes can be constructed. Once these are found, an expression for the velocity eigenmodes is at hand.

Although it is possible to separate the Poincaré equation for rotating triaxial ellipsoids in terms of ellipsoidal coordinates[36], we will adopt a different approach here. We will start from the hypothesis that the inertial modes $\{\mathbf{u}, p, \boldsymbol{\omega}\}$ can be written as polynomials in terms of cartesian coordinates x , y and z . There are multiple motivations for this *ansatz*. First, it is well-known that inertial modes in spheres and spheroids[21, 23, 25] are of this form. Furthermore, polynomial solutions of the Poincaré equation in triaxial ellipsoids have been studied previously, amongst others by Cartan[37] and Lyttleton[36]. Finally, investigations of the linear stability of flows within rotating triaxial ellipsoids [38–41] suggest that the Coriolis operator is closed within certain subspaces of polynomial velocity fields.

This section is now organized as follows. In subsection III A, we present the concept of vector spaces of solenoidal polynomial vector fields. This will allow us, in subsection III B, to reformulate the problem (4)-(6) as a finite-dimensional eigenvalue problem for which a full spectral decomposition is granted. In subsection III C, we will then apply this approach by computing eigenvalues and eigenmodes for subspaces of low polynomial degree. In subsection III D, we will address the first-order viscous corrections of the eigenfrequencies. Finally, subsection III E is devoted to a comparison between theoretical results and numerical simulations.

A. Preliminary concepts

In the following, we present concepts regarding vector spaces of polynomial vector fields in ellipsoidal domains. This theory was originally devised by Gledzer and Ponomarev[38] and Lebovitz [42], and was also thoroughly covered in a recent work by Wu and Roberts [40] (see their Appendix A). It can be described in terms of two vector spaces that we will denote \mathcal{W}_n and \mathcal{V}_n . These are defined as follows: \mathcal{W}_n is the vector space of solenoidal vector fields \mathbf{w} whose cartesian components are homogeneous polynomials of degree n in the cartesian coordinates. \mathcal{V}_n is the vector space of solenoidal vector fields \mathbf{v} that satisfy (6), and are such that $\nabla \times \mathbf{v} \in \mathcal{W}_n$. These vector spaces have the following properties:

1. If we consider a toroidal-poloidal decomposition of an element \mathbf{w} of \mathcal{W}_n :

$$\mathbf{w} = \nabla \times t(\mathbf{r})\mathbf{r} + \nabla \times \nabla \times s(\mathbf{r})\mathbf{r}, \quad (11)$$

then it is always possible to write t as a homogeneous polynomial of degree n and s as a

homogeneous polynomial of degree $n+1$. Furthermore, adopting spherical coordinates (r, θ, ϕ) , we can use the fact that the spherical harmonics $Y_l^m(\theta, \phi)$ of degree l and order m have a polynomial representation[43]. This allows to find an expansion for t and s of the following form:

$$t = r^n \sum_{\substack{l=1 \\ \text{mod}(l,n)=0}}^n t_{lm} Y_l^m(\theta, \phi), \quad (12)$$

$$s = r^{n+1} \sum_{\substack{l=1 \\ \text{mod}(l,n)=1}}^{n+1} s_{lm} Y_l^m(\theta, \phi). \quad (13)$$

To generate a basis for \mathcal{W}_n , one can simply take the vector fields that are associated with each individual spherical harmonic Y_l^m in (12) and in (13).

2. The vector spaces \mathcal{W}_n and \mathcal{V}_n are isomorf, i.e. for any given element \mathbf{w} of \mathcal{W}_n , one can find exactly one solenoidal vector field \mathbf{v} such that $\mathbf{w} = \nabla \times \mathbf{v}$ and \mathbf{v} satisfies the no-penetration condition (6). As such, we can introduce the 'inverse curl operator' $\nabla^{-1} \times$ that acts on elements of \mathcal{W}_n and generates elements of \mathcal{V}_n . We will write this as:

$$\mathbf{v} = \nabla^{-1} \times \mathbf{w}. \quad (14)$$

3. The elements of \mathcal{V}_n are polynomials of degree $n+1$ in the cartesian coordinates.
4. The dimension of \mathcal{W}_n and \mathcal{V}_n is $(n+1)(n+3)$. [42]

For further corroborations, including a more rigorous mathematical underpinning of some of these properties, we refer to Appendix A.

We also introduce the vector spaces $\widetilde{\mathcal{W}}_n$ and $\widetilde{\mathcal{V}}_n$. Here, $\widetilde{\mathcal{W}}_n$ is the vector space of all solenoidal vector fields \mathbf{w} whose cartesian components are polynomials (not necessarily homogeneous) of maximum degree n , i.e.:

$$\widetilde{\mathcal{W}}_n \equiv \mathcal{W}_n \oplus \mathcal{W}_{n-1} \oplus \mathcal{W}_{n-2} \oplus \dots \quad (15)$$

The vector space $\widetilde{\mathcal{V}}_n$ is defined likewise, i.e.:

$$\widetilde{\mathcal{V}}_n \equiv \mathcal{V}_n \oplus \mathcal{V}_{n-1} \oplus \mathcal{V}_{n-2} \oplus \dots \quad (16)$$

In the following, we will denote the dimension of $\widetilde{\mathcal{V}}_n$ and $\widetilde{\mathcal{W}}_n$ as N .

B. The inertial mode equation projected onto $\tilde{\mathcal{V}}_n$ and $\tilde{\mathcal{W}}_n$

We now return to the inertial mode problem, and look for velocity eigenmodes \mathbf{u} that are to $\tilde{\mathcal{V}}_n$; its corresponding vorticity eigenmode $\boldsymbol{\omega}$ belongs to $\tilde{\mathcal{W}}_n$. We use (14) to 'uncurl' expression (7) as follows:

$$\nabla^{-1} \times \nabla \times (2\hat{\mathbf{z}} \times \mathbf{u}) = -i\lambda \nabla^{-1} \times \boldsymbol{\omega}, \quad (17)$$

$$= -i\lambda \mathbf{u}. \quad (18)$$

Now, we note that the operation:

$$\nabla \times (2\hat{\mathbf{z}} \times \mathbf{u}) = -2 \frac{\partial}{\partial z} \mathbf{u} \quad (19)$$

transforms $\mathbf{u} \in \tilde{\mathcal{V}}_n$ into elements of $\tilde{\mathcal{W}}_n$. Indeed, the polynomial degree of \mathbf{u} being part of $\tilde{\mathcal{V}}_n$ is not higher than $n + 1$; thus, $\frac{\partial}{\partial z} \mathbf{u}$ is of maximum polynomial degree n . Moreover, $\frac{\partial}{\partial z} \mathbf{u}$ is solenoidal given that \mathbf{u} also satisfies this property. From these two arguments, it follows that $\nabla \times (2\hat{\mathbf{z}} \times \mathbf{u}) \in \tilde{\mathcal{W}}_n$. This implies that the left-hand side of (18) defines a linear operator \mathcal{L} on $\tilde{\mathcal{V}}_n$. We can recast (18) as follows:

$$\mathcal{L} \mathbf{u} = -i\lambda \mathbf{u}, \quad (20)$$

Furthermore, we endow the vector space $\tilde{\mathcal{V}}_n$ with an inner product that we denote as $\langle \mathbf{v}_j, \mathbf{v}_k \rangle$, and is defined as follows:

$$\langle \mathbf{v}_j, \mathbf{v}_k \rangle = \iiint \mathbf{v}_j^\dagger \cdot \mathbf{v}_k \, dV. \quad (21)$$

Once this is established, it is possible to generate an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_N\}$ for $\tilde{\mathcal{V}}_n$. As such, any element \mathbf{u} of \mathcal{V}_n can uniquely be decomposed as:

$$\mathbf{u} = \sum_{k=1}^N \gamma_k \mathbf{e}_k, \quad (22)$$

with $\gamma_k = \langle \mathbf{u}, \mathbf{e}_k \rangle$.

Of utmost importance now is that the operator \mathcal{L} is skew-Hermitian, i.e.:

$$\langle \mathbf{v}_j, \mathcal{L} \mathbf{v}_k \rangle = -\langle \mathcal{L} \mathbf{v}_j, \mathbf{v}_k \rangle. \quad (23)$$

The proof of this is straightforward. Indeed, one has:

$$\nabla \times \mathcal{L} \mathbf{v}_k = \nabla \times (2\hat{\mathbf{z}} \times \mathbf{v}_k), \quad (24)$$

and thus:

$$\mathcal{L}\mathbf{v}_k = 2\hat{\mathbf{z}} \times \mathbf{v}_k - \nabla\psi. \quad (25)$$

Given the solenoidal character of \mathbf{v}_j and the non-penetration condition (6), it follows that:

$$\langle \mathbf{v}_j, \mathcal{L}\mathbf{v}_k \rangle = \iiint \mathbf{v}_j^\dagger \cdot (2\hat{\mathbf{z}} \times \mathbf{v}_k - \nabla\psi) \, dV = \iiint \mathbf{v}_j^\dagger \cdot (2\hat{\mathbf{z}} \times \mathbf{v}_k) \, dV. \quad (26)$$

In a similar way, one obtains:

$$\langle \mathcal{L}\mathbf{v}_j, \mathbf{v}_k \rangle = \iiint \mathbf{v}_k \cdot (2\hat{\mathbf{z}} \times \mathbf{v}_j)^\dagger \, dV = - \iiint \mathbf{v}_j^\dagger \cdot (2\hat{\mathbf{z}} \times \mathbf{v}_k) \, dV. \quad (27)$$

Comparing (26) and (27), one finds indeed that (23) holds.

Using the expansion (65), the eigenvalue problem (20) can be recast as:

$$-i\lambda\gamma_k = \sum_{l=1}^N \langle \mathbf{e}_k, \mathcal{L}\mathbf{e}_l \rangle \gamma_l. \quad (28)$$

The coefficients $\langle \mathbf{e}_k, \mathcal{L}\mathbf{e}_l \rangle$ define the matrix elements M_{kl} of a square matrix M of finite size that is skew-Hermitian. Although the technique of polynomial vector spaces has been used before to find inertial modes in spheroidal geometry[16], we are the first to substantiate that the ‘inertial mode operator’ can be brought in skew-Hermitian form when projected onto these vector spaces. This is important because, by virtue of the spectral theorem, it ensures the following: (i) The eigenvalues of \mathcal{L} (or M) are purely imaginary (i.e. λ is real) and whether are zero, or come in pairs $\pm i\lambda_k$. (ii) It is possible to find N linearly independent eigenvectors that are solution of the inertial mode equation. Moreover, they are mutually orthogonal with respect to the inner product (21). In other words, the eigenvectors form an orthonormal basis for $\tilde{\mathcal{V}}_n$. These properties hold for any n . As such, it is in principle possible to construct an arbitrarily large set of eigenmodes by using the procedure described above.

C. Linear and quadratic inertial modes

In this subsection, we use the approach outlined in the previous subsections to solve the inertial mode problem within the subspaces $\tilde{\mathcal{V}}_0$ and $\tilde{\mathcal{V}}_1$, i.e. for eigenmodes that are linear, respectively quadratic polynomials in the cartesian coordinates. We will only discuss the final result here. For more details on the intermediate steps leading to these results, we refer to Appendix B.

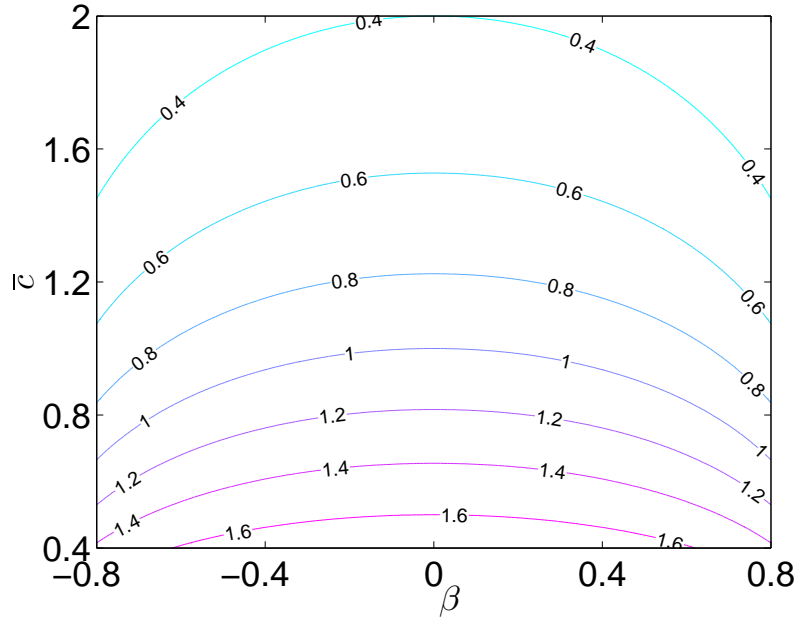


FIG. 1. Isocontours of $|\lambda_{1,2}|$, the eigenfrequency of the spin-over mode given by expression (32), plotted in the (β, \bar{c}) -plane.

For the three-dimensional subspace $\tilde{\mathcal{V}}_0 = \mathcal{V}_0$, we find the following eigenvalues:

$$\lambda_{1,2} = \pm \frac{2ab}{\sqrt{a^2 + c^2}\sqrt{b^2 + c^2}}, \lambda_3 = 0. \quad (29)$$

We can rewrite $\lambda_{1,2}$ in function of two independent, dimensionless parameters \bar{c} and β . To this end, we first introduce R , the mean equatorial radius:

$$R = \sqrt{\frac{a^2 + b^2}{2}}. \quad (30)$$

This allows to define the dimensionless aspect ratio \bar{c} and ellipticity β :

$$\bar{c} = \frac{c}{R}, \quad \beta = \frac{a^2 - b^2}{a^2 + b^2}. \quad (31)$$

As such, we can rewrite $\lambda_{1,2}$ as follows:

$$\lambda_{1,2} = \pm 2 \sqrt{\frac{1 - \beta^2}{(1 + \bar{c}^2 + \beta)(1 + \bar{c}^2 - \beta)}}. \quad (32)$$

In figure 1, we display contours of $|\lambda_{1,2}|$ in the (β, \bar{c}) -plane. The velocity eigenmodes corresponding to these eigenvalues are:

$$\mathbf{u}_{1,2} = \left(\frac{z}{c^2} \hat{\mathbf{x}} - \frac{x}{a^2} \hat{\mathbf{z}} \right) \mp i \frac{b}{a} \sqrt{\frac{a^2 + c^2}{b^2 + c^2}} \left(\frac{z}{c^2} \hat{\mathbf{y}} - \frac{y}{b^2} \hat{\mathbf{z}} \right), \mathbf{u}_3 = -\frac{y}{b^2} \hat{\mathbf{x}} + \frac{x}{a^2} \hat{\mathbf{y}}. \quad (33)$$

All three eigenmodes are of uniform vorticity. For $\mathbf{u}_{1,2}$ the vorticity is purely equatorial; these eigenmodes are usually referred to as the spin-over mode. If we choose $\beta = 0$, we obtain the classical result[25, 27, 41]:

$$\lambda_{1,2} = \pm \frac{2}{\bar{c}^2 + 1} \mathbf{u}_{1,2} = \frac{\bar{z}}{\bar{c}^2} (\hat{\mathbf{x}} \mp i\hat{\mathbf{y}}) - (x \mp iy) \hat{\mathbf{z}}. \quad (34)$$

for the spin-over mode and its eigenfrequency in spheroidal geometry.

We now turn our attention to $\tilde{\mathcal{V}}_1$. Since $\tilde{\mathcal{V}}_0$ is a subspace $\tilde{\mathcal{V}}_1$, the solutions (29)-(33) are also eigensolutions of the inertial mode problem for $\tilde{\mathcal{V}}_1$. The eight other eigenvalues are associated with eigenmodes that are quadratic polynomials. These eigenvalues are the roots of the following characteristic polynomial, which is, for brevity, written in function of $A = a^2/c^2, B = b^2/c^2$:

$$\begin{aligned} & (AB + A + B)[3(A + B) + AB + 8][3(A + 1)B + A + 8B^2][3(B + 1)A + B + 8A^2]\lambda^8 \\ & + 16AB \left[453A^2B^2 + 214A^2B^2(A + B) + 213AB(A^2 + B^2) + 262AB(A + B) + 52AB(A^3 + B^3) \right. \\ & \left. + 11(A^2 + B^2) + 26AB + 55A^3B^3 + 26(A^4 + B^4) + 62(A^3 + B^3) + 18A^2B^2(A^2 + B^2) \right] \lambda^6 \\ & + 32A^2B^2 [36AB(A^2 + B^2) + 52(A^3 + B^3) + 59(A^2 + B^2) \\ & + 12(A + B) + 176AB + 220AB(A + B) + 111A^2B^2] \lambda^4 \\ & + 256A^3B^3 [6(A^2 + B^2) + 19AB + 2(A + B) + 1] \lambda^2 \\ & + 256A^4B^4. \end{aligned} \quad (35)$$

The zeroes of this polynomial come in four pairs $\pm i\lambda_j$ ($j = 1, \dots, 4$). In principle, one can still find an explicit expression for λ_j . These are lengthy and cumbersome, and therefore, are not written here. However, for the spheroidal case (i.e. with $\beta = 0$), the expression is more compact. We find:

$$\lambda_1 = \pm \frac{10+4\sqrt{9+\bar{c}^2}}{11+4\bar{c}^2}, \quad \lambda_2 = \pm \frac{10-4\sqrt{9+\bar{c}^2}}{11+4\bar{c}^2}, \quad \lambda_3 = \pm \frac{2}{\sqrt{1+4\bar{c}^2}}, \quad \lambda_4 = \pm \frac{2}{1+2\bar{c}^2}, \quad (36)$$

a result that was previously obtained by Kerswell[16], who used the same technique as the present one for a spheroidal geometry.

In figure 2, we show curves of constant $|\lambda_j|$ in the (\bar{c}, β) -plane. The isocontours are symmetric with respect to the axis $\beta = 0$, which reflects the fact that the problem is invariant under an exchange of a and b . In figure 3, we illustrate the spatial structures of

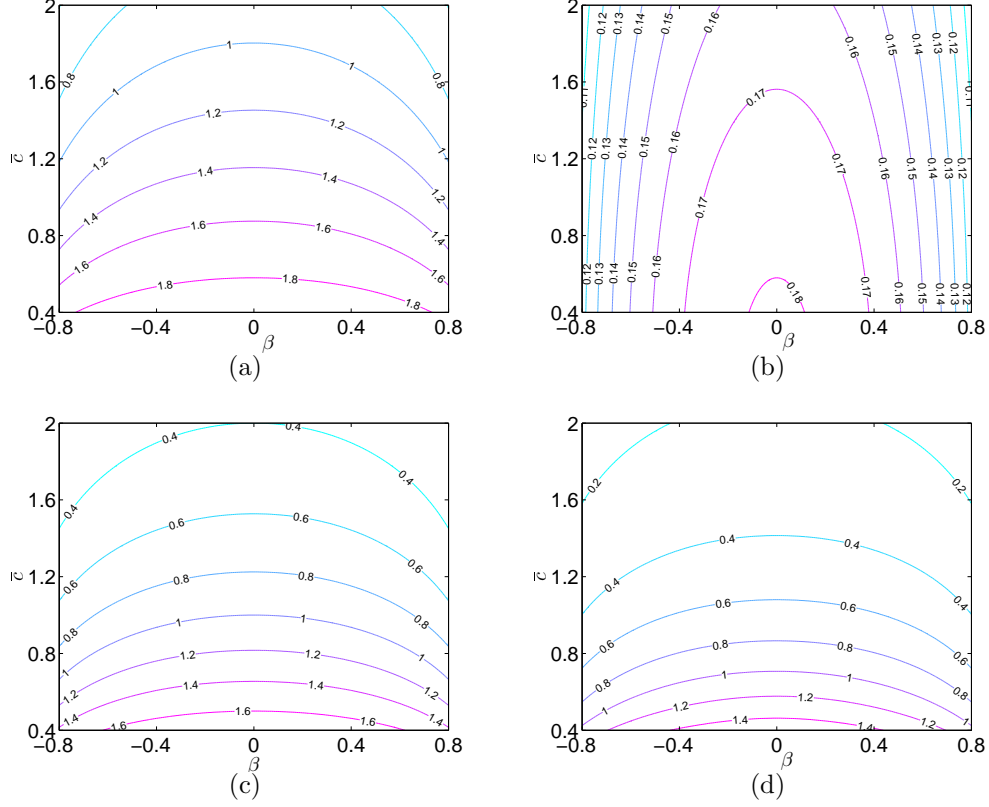


FIG. 2. Isocontours of the absolute value $|\lambda_j|$ of the eigenfrequencies of quadratic inertial modes within the subspace $\tilde{\mathcal{V}}_1$, plotted within the (β, \bar{c}) -plane. Each subfigure corresponds to one pair of roots $\pm i\lambda_j$ of (35). Corresponding circulation patterns of eigenmodes associated with these eigenvalue pairs can be found in figure 3. We note that λ_1 in (36) corresponds to subfigure (a), λ_2 to subfigure (b), etc.

these modes. To do so, we provide meridional circulation patterns of the eigenmodes in the planes $x = 0$ and $y = 0$ for one specific geometry, defined by $\beta = 0.867, \bar{c} = 0.911$. The four values of $|\lambda|$ corresponding to subfigures (a)-(d) are respectively 1.078, 0.09572, 0.5796 and 0.3781. One of the more striking features is that the circulation pattern shown in subfigure (b) is virtually independent from z . Such an almost geostrophic profile is indeed consistent with a low value of $\lambda = 0.09572$. As a further illustration, we also show these patterns for two selected eigenmodes of the vector space $\tilde{\mathcal{V}}_2$, i.e. of polynomials of degree 3, in figure 4.

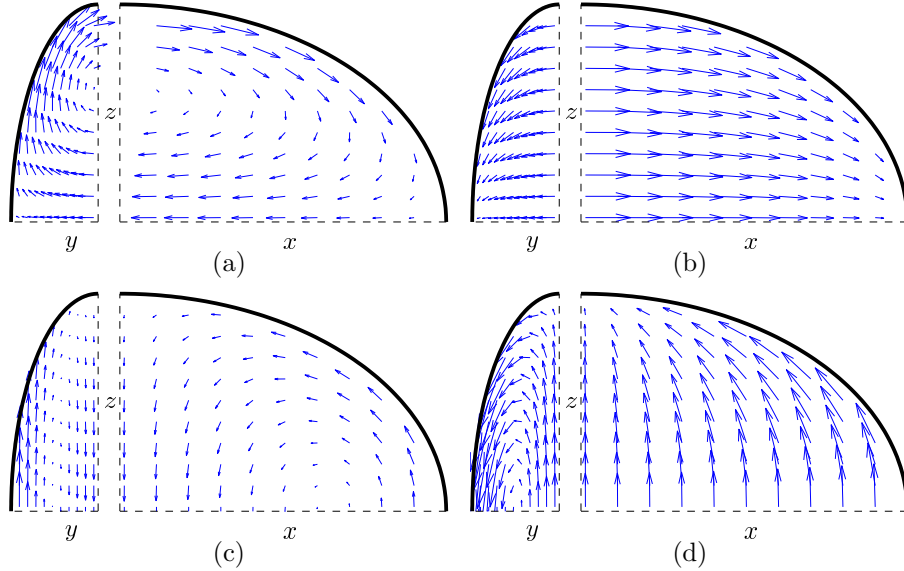


FIG. 3. Meridional circulation patterns of quadratic inertial modes in the planes $x = 0$ and $y = 0$ for an ellipsoid defined by $\beta = 0.867$ and $\bar{c} = 0.911$). Corresponding eigenvalues associated with these eigenmodes can be found in figure 2.

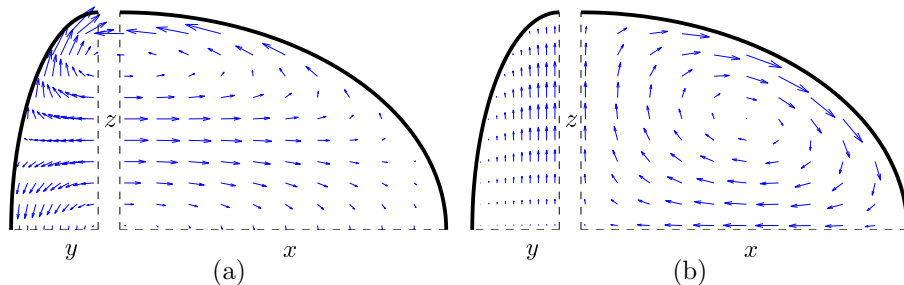


FIG. 4. Meridional circulation patterns of selected cubic inertial modes of $\tilde{\mathcal{V}}_2$ in the planes $x = 0$ and $y = 0$ for an ellipsoid defined by $\beta = 0.867$ and $\bar{c} = 0.911$).

D. Viscous corrections of the spin-over mode frequency

In this section, we are concerned with inertial modes in a viscous fluid. Instead of (5), the governing momentum equation now reads:

$$2\hat{z} \times \mathbf{u} + \nabla p - E\nabla^2 \mathbf{u} = -i\lambda \mathbf{u}, \quad (37)$$

and the boundary condition (6) is replaced by:

$$\mathbf{u} = \mathbf{0}. \quad (38)$$

The parameter E is termed the Ekman number, and obeys the following definition:

$$E = \frac{\nu}{\Omega_0 R^2}. \quad (39)$$

Here ν denotes the kinematic viscosity, which is assumed uniform, and R is defined by (30).

The classical approach to the viscous problem is the one of asymptotic boundary layer theory[23, 25, 27, 32]. For sufficiently small Ekman number, one assumes that the viscous eigenmode can be expanded as the sum of the inviscid eigenmode, and a number of corrections that are asymptotically small in their magnitude and/or spatial extent. More specifically, the velocity field can be decomposed in an interior flow \mathbf{u}_I and a viscous Ekman boundary layer \mathbf{u}_B that allows to accommodate the no-slip condition (38); the wall-normal thickness of this boundary layer is of order-of-magnitude $E^{1/2}$. Each of these contributions, as well as the pressure and eigenfrequency, can then be expanded in in a powers series in the Ekman number:

$$\mathbf{u}_I = \mathbf{u}_{I0} + E^{1/2}\mathbf{u}_{I1} + \dots \quad (40)$$

$$\mathbf{u}_B = \mathbf{u}_{B0} + E^{1/2}\mathbf{u}_{B1} + \dots \quad (41)$$

$$p_I = p_{I0} + E^{1/2}p_{I1} + \dots \quad (42)$$

$$\lambda = \lambda_0 - iE^{1/2}G + \dots, \quad (43)$$

In these expressions, the leading-order terms \mathbf{u}_{I0}, p_{I0} and λ_0 denote the velocity, pressure field and eigenvalue of the corresponding inviscid mode. In general, G is a complex number, i.e. G has both a real and imaginary part, that correspond respectively to a viscous decay rate and frequency shift.

In this work, we are not concerned with solving the structure of the boundary layer flow \mathbf{u}_B or secondary interior flow \mathbf{u}_1 . However, following Greenspan[32] (his section 2.9), we can determine G , merely based on the inviscid frequency and velocity eigenmode profile. Indeed, G is given by:

$$G = -\frac{I_S}{\iiint \mathbf{u}_{I0}^\dagger \cdot \mathbf{u}_{I0} dV}, \quad (44)$$

with I_S the surface integral over the surface of the ellipsoid:

$$I_S = \frac{1}{2^{3/2}} \oint \frac{1}{1 - (\hat{\mathbf{z}} \cdot \hat{\mathbf{n}})^2} \left\{ |\hat{\mathbf{n}} \cdot \hat{\mathbf{z}} \times \mathbf{u}_{I0} - i\hat{\mathbf{z}} \cdot \mathbf{u}_{I0}|^2 |b_+|^{1/2} \left(1 + \frac{ib_+}{|b_+|} \right) + |\hat{\mathbf{n}} \cdot \hat{\mathbf{z}} \times \mathbf{u}_{I0} + i\hat{\mathbf{z}} \cdot \mathbf{u}_{I0}|^2 |b_-|^{1/2} \left(1 + \frac{ib_-}{|b_-|} \right) \right\} dS, \quad (45)$$

and $b_{\pm} = \lambda_0 \pm \hat{z} \cdot \hat{n}$. Although the integral (45) can in general not be computed analytically, I_S and G are readily evaluated numerically. As an illustration, we compute G for the spin-over mode. In figure 5, we show isocontours of the real and imaginary part of G in the (β, \bar{c}) -plane. We find that the real part of G increases in absolute value with decreasing \bar{c} and increasing $|\beta|$. The imaginary part on the other hand also reaches its minimum for $\beta = 0$. Furthermore, we can validate our results by comparing against the expressions for G for oblate spheroidal geometries (i.e. for $\beta = 0$) derived by Zhang and co-workers [27]. Defining the eccentricity $\mathcal{E} = \sqrt{1 - \bar{c}^2}$, these authors provide the following asymptotic formulas for the real and imaginary part of G :

$$\begin{aligned} \Re(G) &= -\frac{3(19 + 9\sqrt{3})}{28\sqrt{2}} + \frac{-1039 + 171\sqrt{3}}{1232\sqrt{2}}\mathcal{E}^2 + \mathcal{O}(\mathcal{E}^4) \\ &= -2.620 - 0.4263\mathcal{E}^2 + \mathcal{O}(\mathcal{E}^4) \end{aligned} \quad (46)$$

$$\begin{aligned} \Im(G) &= -\frac{3(-19 + 9\sqrt{3})}{28\sqrt{2}} + \frac{1039 + 171\sqrt{3}}{1232\sqrt{2}}\mathcal{E}^2 + \mathcal{O}(\mathcal{E}^4) \\ &= 0.2585 + 0.7663\mathcal{E}^2 + \mathcal{O}(\mathcal{E}^4) \end{aligned} \quad (47)$$

In Table I, we compare values of G obtained by numerical evaluation of the expressions (44-45) against their asymptotic counterparts (46-47). We find that the agreement between both values is within one percent for all values of \mathcal{E} considered. However, we observe noticeable discrepancies for the imaginary part for $\mathcal{E} \gtrsim 0.3$. A MATLAB script that allows to compute G for a given β and \bar{c} is provided as online supplementary material.

TABLE I. Comparison between numerically obtained and asymptotic values for G for oblate spheroidal geometries.

\bar{c}	\mathcal{E}	G (from (44 – 45))	G (from (46 – 47))[27]
1	0	$-2.620 + 0.2585i$	$-2.620 + 0.2585i$
0.99	0.14107	$-2.629 + 0.2739i$	$-2.629 + 0.2737i$
0.96	0.28000	$-2.655 + 0.3223i$	$-2.654 + 0.3185i$
0.92	0.39192	$-2.689 + 0.3913i$	$-2.686 + 0.3762i$
0.87	0.49305	$-2.731 + 0.4851i$	$-2.724 + 0.4448i$
0.75	0.66143	$-2.835 + 0.7457i$	$-2.807 + 0.5937i$

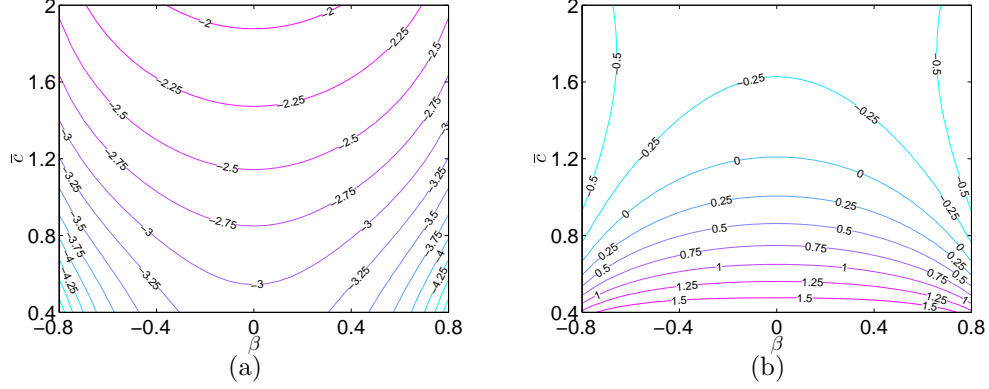


FIG. 5. Isocontours in the (β, \bar{c}) -plane of the real (subfigure a) and imaginary (subfigure b) part of the viscous correction G of spin-over mode eigenfrequency (for the positive value of $\lambda_{1,2}$ in (29)).

E. Numerical validation

To verify the theoretical results obtained in subsection III C, we adopt the following approach. We will compute numerical solutions of the system:

$$\nabla \cdot \mathbf{U} = 0, \quad (48)$$

$$\frac{\partial \mathbf{U}}{\partial \tau} + 2\hat{z} \times \mathbf{U} + \nabla P = E\nabla^2 \mathbf{U} + \mathbf{f}(\mathbf{r}) \cos(\zeta\tau), \quad (49)$$

i.e. the equations of motion for incompressible, rotating, viscous, small-amplitude motion. Our numerical method is based on a finite-volume algorithm for an unstructured set of control volumes[44]. We now prescribe the forcing $\mathbf{f}(\mathbf{r})$ to be of the following form:

$$\mathbf{f}(\mathbf{r}) = C \left(\mathbf{u}_k(\mathbf{r}) + \mathbf{u}_k^\dagger(\mathbf{r}) \right), \quad (50)$$

where \mathbf{u}_k and \mathbf{u}_k^\dagger are a pair of conjugate velocity eigenmodes of $\tilde{\mathcal{V}}_1$, i.e. they are quadratic polynomials. We denote their corresponding eigenvalues as $\pm i\lambda_k$. Furthermore, C is a constant whose exact value is not of major importance. As has been recently argued in the case of libration [45], the harmonic forcing $\mathbf{f}(\mathbf{r}) \cos(\zeta\tau)$ can resonantly drive an inertial mode \mathbf{u}_l , provided the following two conditions are met:

1. The integral $\iiint_V \mathbf{u}_l^\dagger \cdot \mathbf{f}(\mathbf{r}) dV$ does not vanish. From (50) and the orthogonality property of inertial modes (8), it follows that $\mathbf{f}(\mathbf{r})$ can only enter in resonance with \mathbf{u}_k and \mathbf{u}_k^\dagger .

2. The eigenfrequency λ_l and the driving frequency ζ should be (nearly) equal.

The above two conditions imply that resonance can only occur if $\zeta = \pm\lambda_k$. We now verify whether this behavior is recovered in the numerical solution for the choice of parameters $\bar{c} = 1, \beta = 0.45, E = 5 \cdot 10^{-4}$. For this parameter set, we find that one of the eigenmodes associated with the subspace $\tilde{\mathcal{V}}_1$ is characterized by the eigenfrequency $\lambda_k = \pm 0.83236$. This corresponds to subfigure (c) in figure 2, and its meridional circulation pattern is similar to the one shown in figure 3(c). Furthermore, the forcing $\mathbf{f}(\mathbf{r})$ corresponding to this eigenmode pair is:

$$\mathbf{f}(\mathbf{r}) = C (0.82812yz\hat{\mathbf{x}} - 0.48581xz\hat{\mathbf{y}} + 0.312166xy\hat{\mathbf{z}}). \quad (51)$$

In figure 6, we show time-series of the kinetic energy $E_k = \frac{1}{2C^2} \iiint_V |\mathbf{U}|^2 dV$ for different forcing frequencies ζ . We indeed find that resonance only takes place for $\zeta = \lambda_k$. Furthermore, in figure 7, we compare between numerically established profiles of U_z and theoretically obtained solutions for $u_{z,k}$ in the planes $x = 0$ and $y = 0$. We observe an excellent agreement between both profiles, except near the boundaries, where the numerical solution exhibits viscous Ekman layers. Our numerical findings are thus consistent with our theoretical arguments, i.e. the prescribed force $\mathbf{f}(\mathbf{r})$ can resonantly drive the mode \mathbf{u}_k , provided $\zeta = \pm\lambda_k$. As such, our numerical calculations validate the theoretical results obtained in subsection III C.

IV. COMPLETENESS OF THE INERTIAL MODES

One of the major outstanding questions regarding the theory of inertial modes is whether they are complete. By the term complete, we mean that any sufficiently smooth solenoidal velocity field that satisfies the non-penetration condition can be expanded as a sum of inertial modes. This question was first raised by Greenspan in his now-classical monograph on rotating flows[32]. It has essentially remained unanswered until date, apart from a recent work that proved the completeness of inertial modes in a rotating annular duct [46]. The issue of completeness is not without interest, because such a property could provide a new paradigm for the solution of many fluid dynamic problems in rapidly rotating systems. Indeed, since the Coriolis force does not couple different inertial modes, an expansion in terms of inertial modes of the could provide a more efficient way to solve and understand the behaviour of rapidly rotating flows.

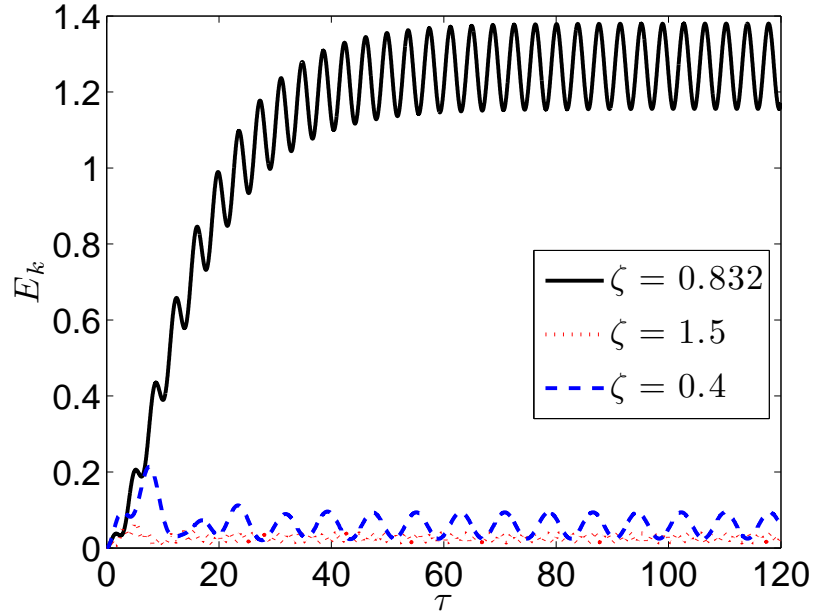


FIG. 6. Time series of the kinetic energy E_k for different forcing frequencies ζ and a spatial force density given by (51).

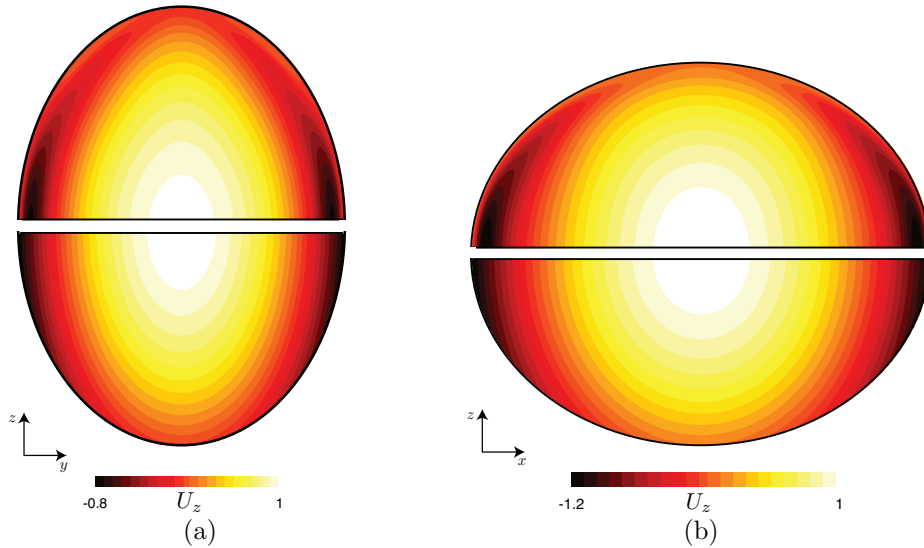


FIG. 7. Isocontours of the numerically established flow U_z (top half) at instant $\tau = 92$, and the theoretically computed z -component $u_{z,k}$ of the inertial mode \mathbf{u}_k (bottom half) in the planes $x = 0$ (subfigure a) and $y = 0$ (subfigure b). Both quantities have been normalized such that their value at the origin is 1.

The approach presented in section III and appendix A now allows us to shed some new light on this outstanding question. As a starting point, we recall that the eigenvalue problem yields a complete basis of eigenvectors for the subspace $\widetilde{\mathcal{V}}_n$. Since $\widetilde{\mathcal{V}}_n$ is isomorf with $\widetilde{\mathcal{W}}_n$, it follows that the vorticity eigenmodes are a basis for $\widetilde{\mathcal{W}}_n$. This means that every solenoidal vector field \mathbf{w} that is polynomial and of maximum degree n , can be expanded as a sum of inertial modes of $\widetilde{\mathcal{W}}_n$. In terms of the Mie representation, the toroidal and poloidal scalars of the vorticity eigenmodes of $\widetilde{\mathcal{W}}_n$ span all scalar functions that are of the form f_t , respectively f_s :

$$f_t(r, \theta, \phi) = \sum_{k=0}^{l+2k \leq n} \sum_{l=1}^n \sum_{m=-l}^l t_{klm} r^{l+2k} Y_l^m(\theta, \phi), \quad (52)$$

$$f_s(r, \theta, \phi) = \sum_{k=0}^{l+2k \leq n+1} \sum_{l=1}^{n+1} \sum_{m=-l}^l s_{klm} r^{l+2k} Y_l^m(\theta, \phi). \quad (53)$$

Taking the limit of $n \rightarrow \infty$, we obtain the following expressions:

$$f_t(r, \theta, \phi) = \sum_{k=0}^{\infty} \sum_{l=1}^{\infty} \sum_{m=-l}^l t_{klm} r^{l+2k} Y_l^m(\theta, \phi), \quad (54)$$

$$f_s(r, \theta, \phi) = \sum_{k=0}^{\infty} \sum_{l=1}^{\infty} \sum_{m=-l}^l s_{klm} r^{l+2k} Y_l^m(\theta, \phi). \quad (55)$$

On the other hand, every function f that is square-integrable over the ellipsoidal volume under consideration, can be expanded as follows[43]:

$$f(r, \theta, \phi) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=-l}^l f_{klm} r^{l+2k} Y_l^m(\theta, \phi). \quad (56)$$

Comparing the above expression to (54) or (55), one finds that they only are different in the lower bound for the index l . Thus, given a square-integrable function f , one can define f^* :

$$f^*(r, \theta, \phi) = f(r, \theta, \phi) - \sum_{k=0}^{\infty} f_{k00} r^{2k}. \quad (57)$$

As such, f^* is of the form (54) or (55), and has thus a decomposition in terms of vorticity eigenmode toroidal or poloidal scalars. Furthermore, we note that:

$$\nabla \times f \mathbf{r} = \nabla \times f^* \mathbf{r}. \quad (58)$$

This suggests that every vorticity field for which both the toroidal and poloidal scalars are square-integrable functions, can be expanded in terms of vorticity eigenmodes. Given

that the subspaces $\tilde{\mathcal{V}}_n$ and $\tilde{\mathcal{W}}_n$ are isomorf, this furthermore suggests that any, sufficiently smooth solenoidal velocity field that satisfies (6) can be expanded on a basis of inertial modes. However, a more rigorous proof is required to show exactly how the regularity and integrability constraints on the Mie scalars carry over to the vorticity field and its corresponding velocity field. Such a proof however, is outside the scope of this work.

V. INTEGRAL PROPERTIES

For inertial modes in rotating spheres and spheroids, it was shown [26, 27] that the following identity holds for any inertial mode \mathbf{u}_l :

$$\langle \mathbf{u}_l, \nabla^2 \mathbf{u}_l \rangle = \iiint \mathbf{u}_l^\dagger \cdot \nabla^2 \mathbf{u}_l dV = 0. \quad (59)$$

We now argue that this integral also vanishes in rotating triaxial ellipsoids. Our line of thought is as follows. Since $\tilde{\mathcal{V}}_{n-2}$ is a subspace of $\tilde{\mathcal{V}}_n$, $\tilde{\mathcal{V}}_n$ has an orthonormal basis of inertial modes $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_N\}$ such that $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\}$ is a basis for $\tilde{\mathcal{V}}_{n-2}$. We recall that, by virtue of the skew-Hermitian nature of the eigenvalue problem of which these modes are the eigenvectors, these (finite-dimensional) bases are complete. We now consider a certain inertial mode \mathbf{u}_l within the subspace $\tilde{\mathcal{V}}_n$ such that \mathbf{u}_l does not belong $\tilde{\mathcal{V}}_{n-2}$, i.e. with $k < l \leq N$. Thus, $\boldsymbol{\omega}_l$ belongs to the subspace $\tilde{\mathcal{W}}_n$, but not to $\tilde{\mathcal{W}}_{n-2}$. This is equivalent to saying that the cartesian components of its associated vorticity field $\boldsymbol{\omega}_l = \nabla \times \mathbf{u}_l$ are polynomials of (exactly) degree n or $n - 1$. Now, $\nabla^2 \mathbf{u}_l$ does not necessarily belong to $\tilde{\mathcal{V}}_{n-2}$. Indeed, nothing guarantees that $\nabla^2 \mathbf{u}_l$ satisfies the non-penetration condition (6). On the other hand, $\nabla^2 \boldsymbol{\omega}_l = \nabla \times \nabla^2 \mathbf{u}_l$ belongs to the subspace $\tilde{\mathcal{W}}_{n-2}$, as it is solenoidal and its polynomial degree is not higher than $n - 2$. Now, we define:

$$\mathbf{Q}_l \equiv \nabla^{-1} \times \nabla^2 \boldsymbol{\omega}_l. \quad (60)$$

By definition, the operator $\nabla^{-1} \times$ transforms elements of $\tilde{\mathcal{W}}_{n-2}$ into elements of $\tilde{\mathcal{V}}_{n-2}$. As such \mathbf{Q}_l belongs to $\tilde{\mathcal{V}}_{n-2}$, and thus satisfies the boundary condition (6), in contrast to $\nabla^2 \mathbf{u}_l$. However, \mathbf{Q}_l and $\nabla^2 \mathbf{u}_l$ are connected by the following identity:

$$\nabla \times \mathbf{Q}_l = \nabla \times \nabla^2 \mathbf{u}_l = \nabla^2 \boldsymbol{\omega}_l, \quad (61)$$

and thus:

$$\nabla^2 \mathbf{u}_l = \mathbf{Q}_l - \nabla \psi. \quad (62)$$

As such, one can write:

$$\langle \mathbf{u}_l, \nabla^2 \mathbf{u}_l \rangle = \langle \mathbf{u}_l, \mathbf{Q}_l \rangle - \langle \mathbf{u}_l, \nabla \psi \rangle = \langle \mathbf{u}_l, \mathbf{Q}_l \rangle. \quad (63)$$

Here, we have used the fact that:

$$\langle \mathbf{u}_l, \nabla \psi \rangle = \iiint \mathbf{u}_l^\dagger \cdot \nabla \psi \, dV = 0. \quad (64)$$

This integral vanishes because \mathbf{u}_l is solenoidal and satisfies the non-penetration condition (6). Now, since $\mathbf{Q}_l \in \tilde{\mathcal{V}}_{n-2}$, we may write it as a sum of inertial modes \mathbf{u}_i of $\tilde{\mathcal{V}}_{n-2}$, i.e.:

$$\mathbf{Q}_l = \sum_{i=1}^k \alpha_i \mathbf{u}_i. \quad (65)$$

This expansion finds its justification in the fact that the inertial modes \mathbf{u}_i of $\tilde{\mathcal{V}}_{n-2}$ are a complete basis for this vector space, as argued at the beginning of this section. This implies that (63) may be recast as:

$$\iiint \mathbf{u}_l^\dagger \cdot \nabla^2 \mathbf{u}_l \, dV = \sum_{i=1}^k \alpha_i \iiint \mathbf{u}_l^\dagger \cdot \mathbf{u}_i \, dV. \quad (66)$$

Hence, since \mathbf{u}_l is orthogonal to every \mathbf{u}_i ($1 \leq i \leq k$), every term in sum on the right-hand side of the above expression vanishes, and we recover (59). We find thus that this remarkable integral property also holds for rotating triaxial ellipsoids.

Furthermore, we find that the integral (66) also vanishes if we replace \mathbf{u}_l^\dagger by \mathbf{u}_m^\dagger , provided that \mathbf{u}_m is an inertial mode that does not belong to $\tilde{\mathcal{V}}_{n-2}$. We can reformulate this as follows. The integral property

$$\iiint \mathbf{u}_m^\dagger \cdot \nabla^2 \mathbf{u}_l \, dV = 0 \quad (67)$$

is satisfied if the highest polynomial order of the cartesian components of \mathbf{u}_m is higher than $n - 2$. This can be identified with another integral property postulated by Liao and Zhang [33]. In their notation, this is written as:

$$\langle \mathbf{u}_{mlM}, \nabla^2 \mathbf{u}_{mlK} \rangle = 0, \quad M \geq K \quad (68)$$

where the highest polynomial degree of \mathbf{u}_{mlK} and \mathbf{u}_{mlM} is $2K + m$, respectively $2M + m$. Thus, the above arguments do extend the integral property discovered by Liao and Zhang [33] to triaxial ellipsoids.

VI. CONCLUDING REMARKS

In this work, we have outlined a procedure to compute inviscid inertial modes in a rotating triaxial ellipsoid. Analytic, explicit expressions for the eigenvalues can be obtained for a limited number of eigenmodes. We have illustrated this for velocity eigenmodes that are linear and quadratic in the cartesian coordinates. Furthermore, it is straightforward to extend our work to polynomial bases of higher degree. Indeed, using a clever combination of state-of-the art numerical and symbolical mathematical software, it is in principle possible to fully automate the procedure described in section III. Taking into account that the dimension of the subspaces increases rapidly with polynomial degree, and that the computational power to solve an eigenvalue problem scales as the third power of its dimension, this may rapidly become very involved. Nevertheless, computational power is so abundant these days that solving the eigenvalue problem up to degree 50 or so should not pose a major problem.

Furthermore, we have shown that two integral properties, originally postulated for inertial modes in spherical and spheroidal domains, also apply to triaxial ellipsoids. These properties reflect that the inertial modes are of polynomial nature. This, at its turn, is closely related to the fact that the solution of the Laplace equation for the gauge function (A20) are ellipsoidal solid harmonics, which have a polynomial representation. In response to the question raised by Liao and Zhang[33], this is really the distinctive feature which makes that these properties hold for ellipsoidal domains, and not e.g. for cylindrical ones. Finally, we have also addressed the question of completeness of inertial modes. Hopefully, our new insight is an impetus for a mathematically rigorous proof of a completeness theorem.

Possible applications of the present work include the linear stability analysis of rotating flows in triaxial ellipsoids in terms of inertial modes. As mentioned in the introduction, previous work have to the limit of small deformation; using the present results, this can be extended to ellipsoids of arbitrary deformation. This in fact comes down to a reinterpretation of the works of Gledzer and Ponomarev [38] Wu and Roberts[40] and Roberts and Wu[47]. We are already undertaking such an effort for the case of flows driven by libration. Finally, a number of issues have remained unaddressed in this work. Although we have computed the leading order viscous correction to the eigenfrequency, one major task to be achieved is the calculation of the viscous Ekman boundary layer. In our understanding, this will necessitate the use of an ellipsoidal coordinate system, and appears a challenging problem.

ACKNOWLEDGMENTS

I am indebted to David Cébron for enlightening and stimulating discussions, and to Elisabeth Canet, Andy Jackson and Jérôme Noir for feedback on first versions of this manuscript. I am thankful to three anonymous reviewers whose comments substantially helped to improve this article. This work was partly funded by ERC grant 247303 ‘‘MFECE’’.

Appendix A: Vector spaces of polynomial solenoidal vector fields

In this section, we discuss the properties of the vector spaces \mathcal{W}_n and \mathcal{V}_n that are presented in subsection IIIII A in more detail:

1. We show that the elements $\mathbf{w} \in \mathcal{W}_n$ have a toroidal-poloidal representation (11) in which t and s are respectively homogeneous polynomials of degree n and $n + 1$. To this end, we start from the well-known identities:

$$\nabla_{\perp}^2 s = \mathbf{r} \cdot \mathbf{w}, \quad (\text{A1})$$

$$\nabla_{\perp}^2 t = \mathbf{r} \cdot \nabla \times \mathbf{w}, \quad (\text{A2})$$

in which ∇_{\perp}^2 denotes the ‘angular momentum operator’:

$$\nabla_{\perp}^2 = -\frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) - \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}. \quad (\text{A3})$$

Since $\mathbf{w} \in \mathcal{W}_n$, the right-hand side of (A1) and (A2) are homogeneous polynomials of degree $n + 1$ and n respectively. Using the fact that the spherical harmonics have a polynomial representation[43], we may recast (A1) and (A2) as follows:

$$\nabla_{\perp}^2 s = r^{n+1} \sum_{\substack{l=1 \\ \text{mod}(l,n)=1}}^{n+1} \sum_{m=-l}^l a_{lm} Y_l^m(\theta, \phi), \quad (\text{A4})$$

$$\nabla_{\perp}^2 t = r^n \sum_{\substack{l=1 \\ \text{mod}(l,n)=0}}^n \sum_{m=-l}^l b_{lm} Y_l^m(\theta, \phi). \quad (\text{A5})$$

The absence of the $l = 0$ harmonic in these expressions is due to the fact that \mathbf{w} and $\nabla \times \mathbf{w}$ are solenoidal. Given that the spherical harmonics are eigenfunctions of the

angular momentum operator, we immediately find:

$$s = r^{n+1} \sum_{\substack{l=1 \\ \text{mod}(l,n)=1}}^{n+1} \sum_{m=-l}^l \frac{a_{lm}}{l(l+1)} Y_l^m(\theta, \phi), \quad (\text{A6})$$

$$t = r^n \sum_{\substack{l=1 \\ \text{mod}(l,n)=0}}^n \sum_{m=-l}^l \frac{b_{lm}}{l(l+1)} Y_l^m(\theta, \phi). \quad (\text{A7})$$

This is indeed of the form (12) and (13) with coefficients $s_{lm} = a_{lm}/(l^2 + l)$ and $t_{lm} = b_{lm}/(l^2 + l)$.

2. The vector spaces \mathcal{V}_n and \mathcal{W}_n are isomorf. To prove this, we start from a toroidal-poloidal decomposition of $\mathbf{v} \in \mathcal{V}_n$:

$$\mathbf{v} = \nabla \times T(\mathbf{r})\mathbf{r} + \nabla \times \nabla \times S(\mathbf{r})\mathbf{r}. \quad (\text{A8})$$

Substituting this into $\mathbf{w} = \nabla \times \mathbf{v}$, with \mathbf{w} of the form (11), leads to:

$$T = s, \quad (\text{A9})$$

$$\nabla^2 S = t. \quad (\text{A10})$$

The Poisson equation (A10) should be supplemented with a boundary condition such that \mathbf{v} satisfies the boundary equation (6). Its general solution is of the form $S = S_P + S_H$, where S_P is a particular solution of (A10) and S_H a harmonic function that allows to satisfy the boundary condition. With t of the form (12), a solution for S_P is readily found:

$$S_P = r^{n+2} \sum_{\substack{l=1 \\ \text{mod}(l,n)=0}}^n \frac{t_{lm}}{(n+3)(n+2) - l(l+1)} Y_l^m(\theta, \phi). \quad (\text{A11})$$

We now consider the following solenoidal vector field:

$$\mathbf{v}_0 = \nabla \times T(\mathbf{r})\mathbf{r} + \nabla \times \nabla \times S_P(\mathbf{r})\mathbf{r}. \quad (\text{A12})$$

It follows that $\mathbf{w} = \nabla \times \mathbf{v}_0$. By virtue of the Helmholtz theorem, we know that any other vector field \mathbf{v} that obeys $\mathbf{w} = \nabla \times \mathbf{v}$ is of the form:

$$\mathbf{v} = \mathbf{v}_0 - \nabla \xi. \quad (\text{A13})$$

To fix the gauge function ξ , we require that \mathbf{v} is solenoidal and satisfies (6). This leads to a Laplace equation for ξ :

$$\nabla^2 \xi = 0, \quad (\text{A14})$$

with boundary condition:

$$\hat{\mathbf{n}} \cdot \nabla \xi = \hat{\mathbf{n}} \cdot \mathbf{v}_0. \quad (\text{A15})$$

Since the right-hand side of (A15) satisfies:

$$\oint \hat{\mathbf{n}} \cdot \mathbf{v}_0 \, dS = \iiint \nabla \cdot \mathbf{v}_0 \, dV = 0, \quad (\text{A16})$$

the problem (A14)-(A15) has a solution that is unique, up to an irrelevant additive constant. Thus for any element \mathbf{w} of \mathcal{W}_n , the corresponding element of \mathcal{V}_n is uniquely determined, and this shows that both vector spaces are indeed connected by an isomorphism.

3. We now also argue that the elements \mathbf{v} of \mathcal{V}_n are polynomials of degree $n + 1$. Using (13), (A9) and (A11), we find that the components of vector field \mathbf{v}_0 defined by (A12) are homogeneous polynomials of degree $n + 1$. The polynomial character of \mathbf{v} now hinges on the nature of the solution for ξ of the problem (A14)-(A15). To this end, we introduce ellipsoidal coordinates (μ_1, μ_2, μ_3) , that are defined as follows [48]:

$$\frac{x^2}{a^2 + \mu_1} + \frac{y^2}{b^2 + \mu_1} + \frac{z^2}{c^2 + \mu_1} = 1, \quad (\text{A17})$$

$$\frac{x^2}{a^2 + \mu_2} + \frac{y^2}{b^2 + \mu_2} + \frac{z^2}{c^2 + \mu_2} = 1, \quad (\text{A18})$$

$$\frac{x^2}{a^2 + \mu_3} + \frac{y^2}{b^2 + \mu_3} + \frac{z^2}{c^2 + \mu_3} = 1. \quad (\text{A19})$$

We can assume without loss of generality, $a > b > c$. Then, the following bounds apply: $\mu_1 > -c^2, -c^2 > \mu_2 > -b^2, -b^2 > \mu_3 > -a^2$. The surfaces of constant μ_1 are confocal ellipsoids; in particular, the surface $\mu_1 = 0$ is the ellipsoidal surface defined by (1) The general solution of (A14) within the interior of the ellipsoid is of the form[48]:

$$\xi(\mu_1, \mu_2, \mu_3) = \sum_{l=0}^{\infty} \sum_{m=1}^{2l+1} A_{lm} E_l^m(\mu_1) E_l^m(\mu_2) E_l^m(\mu_3), \quad (\text{A20})$$

where the functions E_l^m denote Lamé functions of the first kind. It is furthermore well-established that products of the form $E_l^m(\mu_1) E_l^m(\mu_2) E_l^m(\mu_3)$ are polynomials of

degree l in the cartesian coordinates; such products bear the name solid ellipsoidal harmonics. The boundary condition (A15) now allows to determine the coefficients A_{lm} . We can recast (A15) as follows:

$$\left. \frac{\partial \xi}{\partial \mu_1} \right|_{\mu_1=0} = \left(\frac{x}{a^2} \hat{\mathbf{x}} + \frac{y}{b^2} \hat{\mathbf{y}} + \frac{z}{c^2} \hat{\mathbf{z}} \right) \cdot \mathbf{v}_0. \quad (\text{A21})$$

Using (A20), the left-hand side can be further expanded as:

$$\sum_{l=0}^{\infty} \sum_{m=1}^{2l+1} A_{lm} \left. \frac{dE_l^m}{d\mu_1} \right|_{\mu_1=0} E_l^m(\mu_2) E_l^m(\mu_3) = \left(\frac{x}{a^2} \hat{\mathbf{x}} + \frac{y}{b^2} \hat{\mathbf{y}} + \frac{z}{c^2} \hat{\mathbf{z}} \right) \cdot \mathbf{v}_0. \quad (\text{A22})$$

Now, the right-hand side of this expression is a polynomial of degree $n+2$. According to Ferrers[49], this implies that the coefficients A_{lm} are non-zero only if $l \leq n+2$. Since the corresponding solid harmonic functions in (A20) are polynomials of degree l , it follows that the solution for ξ is polynomial of (maximum) degree $n+2$. Hence, the components of \mathbf{v} are of degree $n+1$.

Appendix B: Solution of the eigenvalue problem in $\tilde{\mathcal{V}}_0$ and $\tilde{\mathcal{V}}_1$

In this appendix, we provide more details about the computation of the eigenvalues and eigenmodes of the space $\tilde{\mathcal{V}}_0$ and $\tilde{\mathcal{V}}_1$. The first step required to fulfill this task consists of generating the basis vectors for the respective subspaces. In principle, one could follow the approach sketched in appendix A. However, a more convenient algorithm was presented by Wu and Roberts [40]. They first compute the basis vectors \mathbf{V} for the unit sphere (with respect to coordinates $\mathbf{R} = (X, Y, Z)$), and then use the so-called Poincaré transformation[50]:

$$\mathbf{V} = (V_X, V_Y, V_Z) \rightarrow \mathbf{v} = (v_x, v_y, v_z) = (aV_x, bV_y, cV_z), \quad (\text{B1})$$

$$\mathbf{R} = (X, Y, Z) \rightarrow \mathbf{r} = (x, y, z) = (aX, bY, cZ), \quad (\text{B2})$$

to transform $\mathbf{V}(\mathbf{R})$ into $\mathbf{v}(\mathbf{r})$, where the latter vector field is (i) solenoidal and (ii) satisfies the boundary condition (6) at the surface of the ellipsoid defined by (1) (with respect to the coordinates (x, y, z)). Furthermore, unlike the approach sketched in subsection IIIIB, we will not make use of a set of orthogonal basis vectors. This would present a complication that is unnecessary, as eigenvalues and eigenmodes are invariant under a change of basis.

For the subspace $\tilde{\mathcal{V}}_0 = \mathcal{V}_0$, one has the basis vectors:

$$\mathbf{v}_1 = \left(0, -\frac{z}{c^2}, \frac{y}{b^2} \right), \quad \mathbf{v}_2 = \left(\frac{z}{c^2}, 0, -\frac{x}{a^2} \right), \quad \mathbf{v}_3 = \left(-\frac{y}{b^2}, \frac{x}{a^2}, 0 \right). \quad (\text{B3})$$

Any velocity field \mathbf{u} in this subspace can thus be written as $\mathbf{u} = \sum_{j=1}^3 \gamma_j \mathbf{v}_j$. Substituting this form in the ‘vorticity’ form of the inertial mode equation (7), we obtain three scalar equations (one for each component of the vorticity) for the unknown coefficients γ_j . This can be written under the form:

$$i\lambda\gamma_k = \sum_{j=1}^3 M_{kj}\gamma_j, \quad (\text{B4})$$

with:

$$M = \begin{pmatrix} 0 & \frac{2b^2}{b^2+c^2} & 0 \\ -\frac{2a^2}{a^2+c^2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (\text{B5})$$

The eigenvalues and eigenvectors of this system are then given by (29) and (33).

The subspace $\tilde{\mathcal{V}}_1$ is the direct sum of \mathcal{V}_0 and \mathcal{V}_1 , and is eleven-dimensional. A set of basis vectors for $\tilde{\mathcal{V}}_1$ is found by extending the basis (B3) with the following vectors:

$$\begin{aligned} \mathbf{v}_4 &= \left(0, \frac{xz}{c^2}, -\frac{xy}{b^2}\right), & \mathbf{v}_5 &= \left(-\frac{yz}{c^2}, 0, \frac{xy}{a^2}\right), \\ \mathbf{v}_6 &= \left(zx, zy, c^2 \left(1 - 2\frac{x^2}{a^2} - 2\frac{y^2}{b^2} - \frac{z^2}{c^2}\right)\right), & \mathbf{v}_7 &= \left(-zx, zy, c^2 \left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right)\right), \\ \mathbf{v}_8 &= \left(yx, b^2 \left(\frac{z^2}{c^2} - \frac{x^2}{a^2}\right), -yz\right), & \mathbf{v}_9 &= \left(yx, b^2 \left(1 - 2\frac{x^2}{a^2} - \frac{y^2}{b^2} - 2\frac{z^2}{c^2}\right), yz\right), \\ \mathbf{v}_{10} &= \left(a^2 \left(1 - \frac{x^2}{a^2} - 2\frac{y^2}{b^2} - 2\frac{z^2}{c^2}\right), xy, xz\right), & \mathbf{v}_{11} &= \left(a^2 \left(\frac{y^2}{b^2} - \frac{z^2}{c^2}\right), -xy, xz\right). \end{aligned} \quad (\text{B6})$$

We substitute again $\mathbf{u} = \sum_{j=1}^{11} \gamma_j \mathbf{v}_j$ into (7). This gives rise to three scalar equations, one for each vorticity component ω_k :

$$i\lambda \sum_{j=1}^{11} (N_{kxj}x + N_{kyj}y + N_{kzj}z + N_{k0j})\gamma_j = \sum_{j=1}^{11} (M_{kxj}x + M_{kyj}y + M_{kzj}z + M_{k0j})\gamma_j, \quad (\text{B7})$$

This has to hold for any position (x, y, z) in space, and therefore splits into twelve equations of the form:

$$i\lambda \sum_{j=1}^{11} N_{klj}\gamma_j = \sum_{j=1}^{11} M_{klj}\gamma_j, \quad (\text{B8})$$

where l can each take the values x, y, z or 0 . We recall that the basis vectors $\mathbf{v}_{1,2,3}$ are purely linear in the coordinates. Therefore the coefficients N_{klj} and M_{klj} with $j = 1, 2, 3$ are non-zero only if $l = 0$. Conversely, the other basis vectors do not possess a linear term, and thus N_{k0j} and M_{k0j} are zero for $j = 4, \dots, 11$. This means that the linear system defined by (B7) block-separates into two blocks, one corresponding to $j = 1, 2, 3$, the other one to

to $j = 4, \dots, 11$. The first one is identical to (B4)-(B5); the second one consists of the nine equations:

$$i\lambda \sum_{j=4}^{11} N_{klj} \gamma_j = \sum_{j=4}^{11} M_{klj} \gamma_j, \quad (\text{B9})$$

where l now only takes the values x, y and z . Among these equations, there is one hidden linear dependency. Indeed, since both the left-and right-hand side of (7) are solenoidal by construction, one has the following identities for any $j = 1, 2, \dots, 8$:

$$M_{xxj} + M_{yyj} + M_{zzj} = 0, \quad (\text{B10})$$

$$N_{xxj} + N_{yyj} + N_{zzj} = 0. \quad (\text{B11})$$

In other words, the sum of the three equations for which $k = l$ gives rise to the identity $0 = 0$. One of these equations is thus a linear combination of the other ones, and should thus be left out. If one omits the equation for which $k = l = z$ in (B9), The resulting eigenvalue problem can be written as:

$$i\lambda \sum_{j=4}^{11} N_{Kj} \gamma_j = \sum_{j=4}^{11} M_{Kj} \gamma_j, \quad (\text{B12})$$

with:

$$N = \begin{pmatrix} 1 - \frac{1}{B} & \frac{1}{A} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{B} & -1 - \frac{1}{A} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 - \frac{4}{B} & 1 - \frac{2}{B} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 - \frac{4}{A} & -1 - \frac{2}{A} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 - 2\frac{B}{A} & 1 + 4\frac{B}{A} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 - 2\frac{B}{A} & 1 - 4\frac{B}{A} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 - 4\frac{A}{B} & -1 - 2\frac{A}{B} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 + 4\frac{A}{B} & -1 - 2\frac{A}{B} \end{pmatrix}, \quad (\text{B13})$$

$$M = \begin{pmatrix} 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -8 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 4 & -8 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 2 & 0 & 0 \end{pmatrix}. \quad (\text{B14})$$

The characteristic polynomial of the matrix $N^{-1}M$ is given by (35). It is even and of degree eight. Hence, it is still possible to find explicit solutions for the eigenvalues, i.e. for the roots of this polynomial.

-
- [1] Zhang K. On coupling between the Poincaré equation and the heat equation. *Journal of Fluid Mechanics*. 1994;268:211–229.
 - [2] Sauret A, Cébron D, Le Bars M. Spontaneous generation of inertial waves from boundary turbulence in a librating sphere. *Journal of Fluid Mechanics*. 2013;728.
 - [3] Aldridge KD, Toomre A. Axisymmetric inertial oscillations of a fluid in a rotating spherical container. *Journal of Fluid Mechanics*. 1969;37:307–323.
 - [4] Gans RF. On the precession of a resonant cylinder. *Journal of Fluid Mechanics*. 1970;41(04):865–872.
 - [5] Hollerbach R, Kerswell R. Oscillatory internal shear layers in rotating and precessing flows. *Journal of Fluid Mechanics*. 1995;298:327–339.
 - [6] Noir J, Brito D, Aldridge K, Cardin P. Experimental evidence of inertial waves in a precessing spheroidal cavity. *Geophysical Research Letters*. 2001;28(19):3785–3788.
 - [7] Kerswell R. Tidal excitation of hydromagnetic waves and their damping in the Earth. *Journal of Fluid Mechanics*. 1994;274(-1):219–241.
 - [8] Kelley DH, Triana SA, Zimmerman DS, Tilgner A, Lathrop DP. Inertial waves driven by differential rotation in a planetary geometry. *Geophysical & Astrophysical Fluid Dynamics*. 2007;101(5):469–487.

- [9] Wicht J. Flow instabilities in the wide-gap spherical Couette system. *Journal of Fluid Mechanics*. 2014;738:184–221.
- [10] Le Bars M, Wieczorek MA, Karatekin Ö, Cébron D, Laneuville M. An impact-driven dynamo for the early Moon. *Nature*. 2011;479:215–218.
- [11] Kerswell RR, Malkus WVR. Tidal instability as the source for Io’s magnetic signature. *Geophysical Research Letters*. 1998;25(5):603–606.
- [12] Kerswell RR. Elliptical instability. *Annual Review of Fluid Mechanics*. 2002;34(1):83–113.
- [13] Noir J, Cébron D, Le Bars M, Sauret A, Aurnou J. Experimental study of libration-driven zonal flows in non-axisymmetric containers. *Phys Earth Planet Int*. 2012;204-205:1–10.
- [14] Cébron D, Le Bars M, Noir J, Aurnou JM. Libration driven elliptical instability. *Physics of Fluids*. 2012;24:061703.
- [15] Wu CC, Roberts PH. On a dynamo driven topographically by longitudinal libration. *Geophys Astrophys Fluid Dyn*. 2013;107:1–25.
- [16] Kerswell R. The instability of precessing flow. *Geophysical & Astrophysical Fluid Dynamics*. 1993;72(1):107–144.
- [17] Le Bars M, Lacaze L, Le Dizès S, Le Gal P, Rieutord M. Tidal instability in stellar and planetary binary systems. *Physics of the Earth and Planetary Interiors*. 2010;178(1-2):48–55.
- [18] Lacaze L, Le Gal P, Le Dizès S. Elliptical instability in a rotating spheroid. *Journal of Fluid Mechanics*. 2004;505:1–22.
- [19] Kelvin L. On the oscillations of a columnar vortex. *Phil Mag*. 1880;10:155–168.
- [20] Poincaré H. Sur l’équilibre d’une masse fluide animée d’un mouvement de rotation. *Acta Mathematica*. 1885;7:259–380.
- [21] Bryan G. The waves on a rotating liquid spheroid of finite ellipticity. *Philosophical Transactions of the Royal Society of London A*. 1889;180:187–219.
- [22] Bjerknes V, Bjerknes J, Solberg H, Bergeron T. *Physikalische Hydrodynamik*. vol. 1. Berlin, Germany: J. Springer; 1934.
- [23] Greenspan H. On the transient motion of a contained rotating fluid. *Journal of Fluid Mechanics*. 1964;20(04):673–696.
- [24] Roberts P, Stewartson K; Cambridge Univ Press. On the motion of a liquid in a spheroidal cavity of a precessing rigid body. II. *Mathematical Proceedings of the Cambridge Philosophical Society*. 1965;61(01):279–288.

- [25] Kudlick MD. On transient motions in a contained, rotating fluid. Massachusetts Institute of Technology. Cambridge, MA, United States; 1966.
- [26] Zhang K, Earnshaw P, Liao X, Busse F. On inertial waves in a rotating fluid sphere. *Journal of Fluid Mechanics*. 2001;437:103–119.
- [27] Zhang K, Liao X, Earnshaw P. On inertial waves and oscillations in a rapidly rotating spheroid. *Journal of Fluid Mechanics*. 2004;504:1–40.
- [28] Rieutord M, Valdettaro L. Inertial waves in a rotating spherical shell. *Journal of Fluid Mechanics*. 1997;341:77–99.
- [29] Rieutord M, Georgeot B, Valdettaro L. Inertial waves in a rotating spherical shell: attractors and asymptotic spectrum. *Journal of Fluid Mechanics*. 2001;435:103–144.
- [30] Baruteau C, Rieutord M. Inertial waves in a differentially rotating spherical shell. *Journal of Fluid Mechanics*. 2013;719:47–81.
- [31] Maas LRM. On the amphidromic structure of inertial waves in a rectangular parallelepiped. *Fluid Dynamics Research*. 2009;33:373–401.
- [32] Greenspan HP. *The Theory of Rotating Fluids*. Cambridge University Press; 1968.
- [33] Liao X, Zhang K. A New Integral Property of Inertial Waves in Rotating Fluid Spheres. *Proceedings of the Royal Society of London A Mathematical and Physical Sciences*. 2009;465(2104):1075–1091.
- [34] Maas LRM, Benielli D, Sommeria J, Lam FPA. Observation of an internal wave attractor in a confined, stably stratified fluid. *Planets*. 1994;369:628–630.
- [35] Liao X, Zhang K. Inertial oscillation, inertial wave and initial value problem in rotating annular channels. *Geophysical & Astrophysical Fluid Dynamics*. 2009;103(2-3):199–222.
- [36] Lyttleton RA. *The stability of rotating liquid masses*. Cambridge University Press; 1953.
- [37] Cartan M. Sur les petites oscillations d’une masse fluide. *Bull Sci Math*. 1922;46:317–352.
- [38] Gledzer E, Ponomarev V. Instability of bounded flows with elliptical streamlines. *Journal of Fluid Mechanics*. 1992;240(-1):1–30.
- [39] Lebovitz NR, Lifschitz A. New global instabilities of the Riemann ellipsoids. *The Astrophysical Journal*. 1996;458:699.
- [40] Wu CC, Roberts PH. High order instabilities of the Poincaré solution for precessionally driven flow. *Geophysical & Astrophysical Fluid Dynamics*. 2011;105(2-3):287–303.
- [41] Kerswell R. Elliptical instabilities of stratified, hydromagnetic waves. *Geophysical & Astro-*

- physical Fluid Dynamics. 1993;71(1):105–143.
- [42] Lebovitz NR. The stability equations for rotating, inviscid fluids: Galerkin methods and orthogonal bases. *Geophysical & Astrophysical Fluid Dynamics*. 1989;46(4):221–243.
- [43] Backus G. *Foundations of Geomagnetism*. Cambridge University Press; 1996.
- [44] Vantieghem S. Numerical simulations of quasi-static magnetohydrodynamics using an unstructured finite-volume solver: development and applications. Université Libre de Bruxelles; 2011.
- [45] Zhang K, Chan KH, Liao X, Aurnou JM. The non-resonant response of fluid in a rapidly rotating sphere undergoing longitudinal libration. *J Fluid Mech*. 2013;720:212–236.
- [46] Cui Z, Zhang K, Liao X. On the completeness of inertial wave modes in rotating annular channels. *Geophys Astrophys Fluid Dyn*. 2014;108(1):44–59.
- [47] Roberts PH, Wu CC. On flows having constant vorticity. *Physica D: Nonlinear Phenomena*. 2011;240:1615–1628.
- [48] Whittaker ET, Watson GN. *A Course in Modern Analysis*, 4th ed. Cambridge Univ. Press; 1990.
- [49] Ferrers NM. *An elementary treatise on spherical harmonics and subjects connected with them*. London, UK: Macmillan and co.; 1877.
- [50] Poincaré R. Sur la précession des corps déformables. *Bull Astr*. 1910;27:321.