Applied Mathematics and Computation 235 (2014) 17–25

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Numerical approach to the Stokes problem with high contrasts in viscosity

I.S. Lobanov [b], I.Yu. Popov [b,*], A.I. Popov [b], T.V. Gerya [a]

[a] Institute of Geophysics, Department of Earth Sciences, Swiss Federal Institute of Technology Zurich (ETH), 5 Sonnegstrasse, CH-8092 Zurich, Switzerland
[b] St. Petersburg National Research University of Information Technologies, Mechanics and Optics, 49 Kronverkskiy, St. Petersburg 197101, Russia

ARTICLE INFO

Keywords:
Stokes flow
Variable viscosity
Finite-differences approach

ABSTRACT

An algorithm based on Sherman–Morrison–Woodbury formula to solve numerically the Stokes problem is described. The algorithm allows to obtain the solution for viscosity contrasts up to ten orders of magnitude, moreover solution speed does not depend on viscosity contrast. Tests of accuracy of the algorithm are provided.

© 2014 Elsevier Inc. All rights reserved.

1. Introduction

The problem of high viscosity contrasts is one of the computationally challenging problems in geodynamic modeling as the fluid viscosity changes abruptly at the interface between two domains of different viscosity, and the use of standard numerical techniques leads sometimes to significant numerical errors at and in the vicinity of the interfaces (see, e.g., [1–3]). The problem of numerical handling of density and viscosity contrasts in geodynamic modeling were intensively studied in 80–90s of the XXth century by Christensen [4,5], Lenardic and Kaula [6], Naimark and Ismail-Zadeh [7,8], and Naimark et al. [9].

In simulation of two phase flows viscosity jumps are often supported on rather small sets [10]. In mathematical physics and PDE theory perturbations supported on a small set are well known under names point perturbations, finite rank perturbations and so on; first publications on the subject dates back to 1930-s, see e.g. [11]. The modern formulation of finite rank perturbations is based on the Krein operator extension theory, boundary triples and similar techniques, and allows to quite easily calculate resolvent for perturbations of partial-differential operator supported on curves, surfaces and fractals (see, e.g., [12–14] and references therein). By authors opinion inclusion of the Stokes operator into the above mentioned framework can be very fruitful, but this question will be discussed somewhere else. Few first steps in this direction were made for 2D Stokes equations with constant viscosity (see [15–17]).

In numerical computations Krein formula has its analog called Sherman–Morrison–Woodbury formula, which allows to compute inverse of a small rank perturbation to a matrix [18–21]. We propose to apply the formula to the operator for Stokes and continuity equations decomposed to two addenda: one can be inverted fast numerically using constant viscosity solver and the other is of rank proportional to viscosity jump support. The Woodbury formula allows to obtain explicit solution for variable viscosity problem, however the solution is cumbersome even for small rank perturbations. In the article we propose algorithm that can be used to obtain the solution numerically. The described method is advantageous in terms of computation speed, which does not depend on scale of viscosity jumps.

* Corresponding author.
E-mail addresses: popov1955@gmail.com (I.Yu. Popov), taras.gerya@erdw.ethz.ch (T.V. Gerya).

http://dx.doi.org/10.1016/j.amc.2014.02.084
0096-3003/© 2014 Elsevier Inc. All rights reserved.

2. Physical problem

The flow of geomaterials over geological timescales is calculated by solving the momentum equation neglecting inertial terms (Stokes equations) [22,23]

$$\frac{\partial \sigma_{ij}}{\partial x_j} = -\rho g_i. \tag{1}$$

The Stokes equations for the incompressible viscous fluid describe the balance between the external body forces and viscous stresses. The viscous force is formulated as the gradient of the stress tensor $\boldsymbol{\sigma}$ and the body force is written as the product of the fluid density ρ and the gravitational acceleration vector $\boldsymbol{g}$. Moreover, in the absence of melting and phase transitions, geodynamic flows are considered as incompressible. Incompressibility is enforced by coupling the aforementioned equations with the continuity equation

$$\frac{\partial v_i}{\partial x_i} = 0, \tag{2}$$

where $\boldsymbol{v}$ is the velocity vector and x_i is a spatial coordinate. Eqs. (1) and (2) are valid over the model domain which we denote by Ω. The developed below method imposes no restriction on dimension of space $\mathbb{R}^d$ containing Ω; however for the sake of simplicity below we focus only on two dimensional case.

To close the system (2), the equations for the conservation of momentum and mass are supplemented with two boundary conditions. Decomposing the boundary of Ω into two non-overlapping regions, denoted by $\partial\Omega_N$ and $\partial\Omega_D$, the boundary conditions are written as

$$\sum_{j=1}^{d} \sigma_{ij} n_j = a_i, \quad \boldsymbol{x} \in \partial\Omega_N, \quad i = 1, \ldots, d \tag{3}$$

and

$$\boldsymbol{v} = \boldsymbol{b}, \quad \boldsymbol{x} \in \partial\Omega_D. \tag{4}$$

Here $\boldsymbol{n}$ is the outward point normal to the boundary of Ω, $\boldsymbol{a}$ is an applied traction and $\boldsymbol{b}$ is a prescribed velocity.

The mechanical behavior of the material is defined by a constitutive relationship. We relate the stress tensor $\boldsymbol{\sigma}$ to the strain rate tensor $\boldsymbol{\epsilon}$, using a linear, isotropic viscous rheology given by

$$\sigma_{ij} = -p\delta_{ij} + 2\eta\dot{\epsilon}_{ij}, \tag{5}$$

where δ is the Kronecker delta, p is the pressure, η is the viscosity and the strain rate is given by

$$\dot{\epsilon}_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right). \tag{6}$$

Prior to any discretization, the equations above are nondimensionalized by means of dynamic scaling. This scaling is achieved by first defining a set of characteristic units such as a characteristic length (e.g., domain size), a characteristic time (e.g., inverse background strain rate), a characteristic viscosity (e.g., minimum viscosity in the domain) and secondly deriving all the related characteristic units (mass, stress, force...). We employ characteristic units that are equal to 1. The results are not scaled back to dimensional units and therefore the velocity errors and pressure are dimensionless.

3. Discretization

We solve Eqs. (1) and (2) for the primitive variables v_i and p. The governing equations can be written as follows

$$\sum_{j=1}^{d} \frac{\partial}{\partial x_j}\left(\eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\right) - \frac{\partial p}{\partial x_i} = -\rho g_i, \quad i = 1, \ldots, d. \tag{7}$$

$$\sum_{i=1}^{d} \frac{\partial v_i}{\partial x_i} = 0. \tag{8}$$

We use finite-difference discretization on non-staggered grid:

$$\begin{cases} h_j^{-1}\nabla_j\left(\eta\left(h_j^{-1}\Delta_j v_i + h_i^{-1}\Delta_i v_j\right)\right) + \sum_{j\neq i} h_j^{-1}\Delta_j\left(\eta\left(h_j^{-1}\nabla_j v_i + h_i^{-1}\nabla_i v_j\right)\right) - h_i^{-1}\nabla_i p = -\rho g_i, \\ \sum_{i=1}^{d} h_i^{-1}\Delta_i v_i = 0, \end{cases} \tag{9}$$

where h_i is a mesh size, Δ_i is a forward difference, ∇_i is a backward difference,

$$\Delta_i f(x_1,\dots,x_d) = f(x_1,\dots,x_i+h_i,\dots,x_d) - f(x_1,\dots,x_i,\dots,x_d),$$

$$\nabla_j f(x_1,\dots,x_d) = f(x_1,\dots,x_i,\dots,x_d) - f(x_1,\dots,x_i-h_i,\dots,x_d).$$

4. Arrow-Hurwicz algorithm

Eq. (9) can be written in the form:

$$L\begin{pmatrix}\boldsymbol{v}\\ p\end{pmatrix} = \begin{pmatrix}-\rho\boldsymbol{g}\\ 0\end{pmatrix}, \tag{10}$$

where $\boldsymbol{v}$ is the velocity field, and the operator L acting in the space of described above discretizations of velocity and pressure fields is defined by its matrix:

$$L = \begin{pmatrix}S & D\\ D^* & 0\end{pmatrix}, \tag{11}$$

where submatrices S and D are defined by its action on velocity and pressure fields:

$$(S\boldsymbol{v})_i = h_j^{-1}\nabla_j\left(\eta\left(h_j^{-1}\Delta_j v_i + h_i^{-1}\Delta_i v_j\right)\right) + \sum_{j\neq i} h_j^{-1}\Delta_j\left(\eta\left(h_j^{-1}\nabla_j v_i + h_i^{-1}\nabla_i v_j\right)\right), \tag{12}$$

$$(Dp)_i = h_i^{-1}\nabla_i p, \quad D^*\boldsymbol{v} = \sum_i h_i^{-1}\nabla_i v_i. \tag{13}$$

The equation belongs to saddle-point problems (see [18, Chapter 8.4]). The solution of Eq. (10) can be obtained by an iterative method as the limit $(\boldsymbol{v},p)^T$ of approximations $(\boldsymbol{v}_k,\boldsymbol{p}_k)$. e.g. by a variant of Arrow-Hurwitz algorithm,

$$\boldsymbol{v} = \lim_{k\to\infty}\boldsymbol{v}^k, \quad p = \lim_{k\to\infty} p^k, \tag{14}$$

$$\boldsymbol{v}^{k+1} = \boldsymbol{v}^k + \epsilon\left(-\rho\boldsymbol{g} + S\boldsymbol{v}^k - Dp^k\right), \quad p^{k+1} = p^k + \omega D^*\boldsymbol{v}^k. \tag{15}$$

Unfortunately, the convergence of the iteration method is not guaranteed for arbitrary viscosity η, and choice of the relaxation parameters ϵ,ω is complicated. However assuming that η is constant, the solution can be found very fast using e.g. a multi-grid method or the Fourier transform.

5. Multi-grid method

The multi-grid method can be used to solve Eq. (9) roughly in linear time with respect to size of the grid. Here we briefly recall the method (see [19, Chapter 20.6], and [20, Chapter 7.6]). Convergence of the method and other details are described in [24].

In the section we work with several grids, therefore we introduce notation L_n for the operator L in (10) on nth grid. We assume that $(n+1)$th grid is coarse than the grid n. Let R_n be a restriction operator, which is a mapping from the fine grid n to the coarse one $n+1$, and let S_n be a smoothing operator, which maps a function on the coarse grid $n+1$ to a function on the grid n. A common choice for R_n and S_n is the bilinear smoothing.

We are going to solve the Eqs. (10) on the most coarse grid, which is written as

$$L_0\boldsymbol{X} = \boldsymbol{Y}, \tag{16}$$

denoting

$$\boldsymbol{X} = \begin{pmatrix}\boldsymbol{v}\\ p\end{pmatrix}, \quad \boldsymbol{Y} = \begin{pmatrix}-\rho\boldsymbol{g}\\ 0\end{pmatrix}.$$

Let P_0 be a precondition for the Eq. (16).

The considered multi-grid algorithm consists of the following steps:

1. Fix an approximation $\boldsymbol{X}_0^0$ to the solution $\boldsymbol{X}$ of Eq. (16). An appropriate choice is $\boldsymbol{X}_0^0 = P_0\boldsymbol{Y}$.
2. Compute the residual $\boldsymbol{\rho}_0^k = L_n\boldsymbol{X}_0^k - \boldsymbol{Y}$, where k is the iteration step.
3. Compute an approximate solution $\boldsymbol{A}_n^k$ of the equation $L_0\boldsymbol{A}_0^k = \boldsymbol{\rho}_0^k$, which is a correction to an approximate solution $\boldsymbol{X}_0^k$. The approximate solution is found solving the equation consequently on several coarser grids. First compute the restriction of the residual to the coarsest grid,

$$\boldsymbol{\rho}_n^k = R_{n-1}\cdots R_0\boldsymbol{\rho}_0^k.$$

4. Find an approximate solution $\boldsymbol{A}_n^k$ to the equation

$$L_n\boldsymbol{A}_n^k = \boldsymbol{\rho}_n^k \tag{17}$$

Taking several iteration by Jacobi or Gauss–Seidel methods is enough.

5. Smooth $\boldsymbol{A}_m^k$ to the finer grid

$$\boldsymbol{B}_{m-1}^k = S_{m-1}\boldsymbol{A}_m^k,$$

where m is the last worked out grid.

6. Solve the equation

$$L_{m-1}\boldsymbol{C}_m^k = L_{m-1}\boldsymbol{B}_{m-1}^k - \boldsymbol{\rho}_n^k. \tag{18}$$

Again one can stop after several Jacobi or Gauss–Seidel iterations.

7. Set

$$\boldsymbol{A}_{m-1}^k = \boldsymbol{B}_{m-1}^k - \boldsymbol{C}_{m-1}^k.$$

8. Repeat steps 5–8 until the solution $\boldsymbol{A}_0^k$ on the finest grid is obtained.
9. The next approximation is

$$\boldsymbol{X}_0^{k+1} = \boldsymbol{X}_0^k - \boldsymbol{A}_0^k.$$

10. Repeat steps 2–10 until desired precision is obtained.

If only one grid is used ($n = 0$), and the auxiliary Eq. (17) is solved by Jacobi iterations, then the whole algorithm is again Jacobi iterations. A good choice of approximations to solutions of (18) can speed up convergence significantly, however this choice is very tricky and easily can ruin convergence.

6. Operator splitting

The multi-grid method recalled in the previous section can be used for very fast calculation of the solution of Eq. (10) for a constant viscosity η. If variation of η is small, the multi-grid iterations converge slower, but still the method converges fast enough. Further we show, how to get the solution in a reasonable time, if η has large jump in a few points.

We write viscosity as a sum

$$\eta = \eta_r + \eta_s,$$

where the gradient of η_r vanishes on most of the domain and can have arbitrary large values in a small region. The Stokes operator S is decomposed to the sum

$$S = S_r + S_s,$$

where

$$(S_k\boldsymbol{v})_i = h_j^{-1}\nabla_j\left(\eta_k\left(h_j^{-1}\Delta_j v_i + h_i^{-1}\Delta_i v_j\right)\right) + \sum_{j\neq i} h_j^{-1}\Delta_j\left(\eta_k\left(h_j^{-1}\nabla_j v_i + h_i^{-1}\nabla_i v_j\right)\right), \quad k = r, s. \tag{19}$$

We suppose that S_r can be inverted somehow, e.g. by multi-grid method.

Denote by T_i the translation operator,

$$T_i f(x_1,\ldots,x_d) = f(x_1,\ldots,x_i + h_i,\ldots,x_d).$$

Using the identity

$$\Delta_i(ab) = (\Delta_i a)b + (T_i a)(\Delta_i b) = (\Delta_i a)(T_i b) + a(\Delta_i b), \tag{20}$$

we write the operator S_s as the sum

$$S_s = S_a + S_p,$$

$$(S_a\boldsymbol{v})_i = \eta_r h_j^{-1}\left(h_j^{-1}\nabla_j\Delta_j v_i + h_i^{-1}\nabla_j\Delta_i v_j\right) + \sum_{j\neq i}\eta_r h_j^{-1}\left(h_j^{-1}\Delta_j\nabla_j v_i + h_i^{-1}\Delta_j\nabla_i v_j\right), \tag{21}$$

$$(S_p\boldsymbol{v})_i = h_j^{-1}(\nabla_j\eta_r)T_j\left(h_j^{-1}\Delta_j v_i + h_i^{-1}\Delta_i v_j\right) + \sum_{j\neq i} h_j^{-1}(\Delta_j\eta_r)T_j\left(h_j^{-1}\nabla_j v_i + h_i^{-1}\nabla_i v_j\right). \tag{22}$$

The operator S_a is a symmetric diagonal dominated operator, which is only by a multiplier differs from the Stokes operator with a constant viscosity, hence it can be easily inverted using an iterative technique. The addendum S_p is a non-symmetric first order operator with large off-diagonal coefficients, which ruins convergence of iteration schemes for S. Our proposal is to consider S as an additive perturbation of the operator $S_0 := S_r + S_a$ by the operator S_p, where the rank of S_p is much smaller than rank of S_0.

The whole operator L of the system (10) is also decomposed according to the decomposition of S:

$$L = L_0 + L_p, \quad L_0 = \begin{pmatrix} S_0 & D \\ D^\star & 0 \end{pmatrix}, \quad L_p = \begin{pmatrix} S_p & 0 \\ 0 & 0 \end{pmatrix}.$$

7. Woodbury formula

An analogue of the Krein formula in the discrete case is the so called Woodbury formula, see in [19, Chapter 2.7.3]. Other applications of the formula, which are not covered here, can be found in [21].

Suppose that A is a $N \times N$ matrix, while U and V are $N \times M$ matrices with $M < N$. The Woodbury formula allows to compute inverse to the matrix A perturbed by UV^T as a correction to the inverse of A:

$$\left(A + UV^T\right)^{-1} = A^{-1} - A^{-1}U\left(E + V^TA^{-1}U\right)^{-1}V^TA^{-1}, \tag{23}$$

where E is the identity matrix. By the formula, the inverse of $N \times N$ perturbed matrix $A + UV^T$ can be obtained by inversion only $M \times M$ matrix $E + V^TA^{-1}U$, provided the inverse of the unperturbed matrix A is known.

Since our aim is to solve an equation instead of calculating the inverse matrix, we use the formula as follows. Choose the column matrices $\boldsymbol{u}_k, \boldsymbol{v}_k$, $k = 1, .., r$, $r :=$ rank UV^T, such that

$$UV^T = \sum_k \boldsymbol{u}_k \boldsymbol{v}_k^T.$$

Our purpose is to solve the following equation for every given $\boldsymbol{b}$:

$$\left(A + \sum_k \boldsymbol{u}_k \boldsymbol{v}_k^T\right)\boldsymbol{x} = \boldsymbol{b}. \tag{24}$$

To find the solution the following steps are required:

1. Solve r auxiliary problems,

   $$A\boldsymbol{z}_k = \boldsymbol{u}_k,$$

and construct the matrix Z by columns of the obtained solutions $\boldsymbol{z}_k$, $Z = (\boldsymbol{z}_1, \ldots, \boldsymbol{z}_r)$.

2. Do the matrix inversion

   $$H = \left(E + V^TZ\right)^{-1},$$

where E is the identity operator.

3. Solve the one further auxiliary problem,

   $$A\boldsymbol{y} = \boldsymbol{b}.$$

4. The desired solution is given by

   $$\boldsymbol{x} = \boldsymbol{y} - ZHV^T\boldsymbol{y}.$$

8. Application of Woodbury formula to Stokes-continuity equation

Consider the operator $L = L_0 + L_p$. Denote by n the total number of the grid points and by r the number of points, where the gradient $\Delta_j\eta_s$ does not vanish. Due to Eq. (22), the operator S_p has rank less or equals to $r \cdot d$, where d is the space dimension. By definition, rank of L_p coincides with rank of S_p. Normally we want r to be much smaller than total number n of cells of the grid. Let $A = L_0, U$ be the identity operator E, and $V = L_p$. Then we can apply formula (23), inverting L_0 by a fast iterative technique, such as multi-grid. The cost of the formula application consists of:

1. $r + 1$ times solution of the equation $L_0\boldsymbol{x} = \boldsymbol{y}$ for distinct $\boldsymbol{y}$. The complexity of the procedure depends on the method and can varies from $O(n^d)$ to $O(n)$.
2. Inversion of $r \times r$ matrix, which requires less than $O(r^3)$ operations.

Clearly, for large r the cost of calculation of $H = (E + V^T Z)^{-1}$ may be bigger than direct inversion of S. However, normally the method has the following two advantages:

1. The solution can be obtained for sure whatever η_s is given. There is no adjustable parameters, neither need to guess.
2. The convergence of S_0 is generally better than one of S_p, hence for small r the application of Woodbury formula is faster than traditional multi-grid.

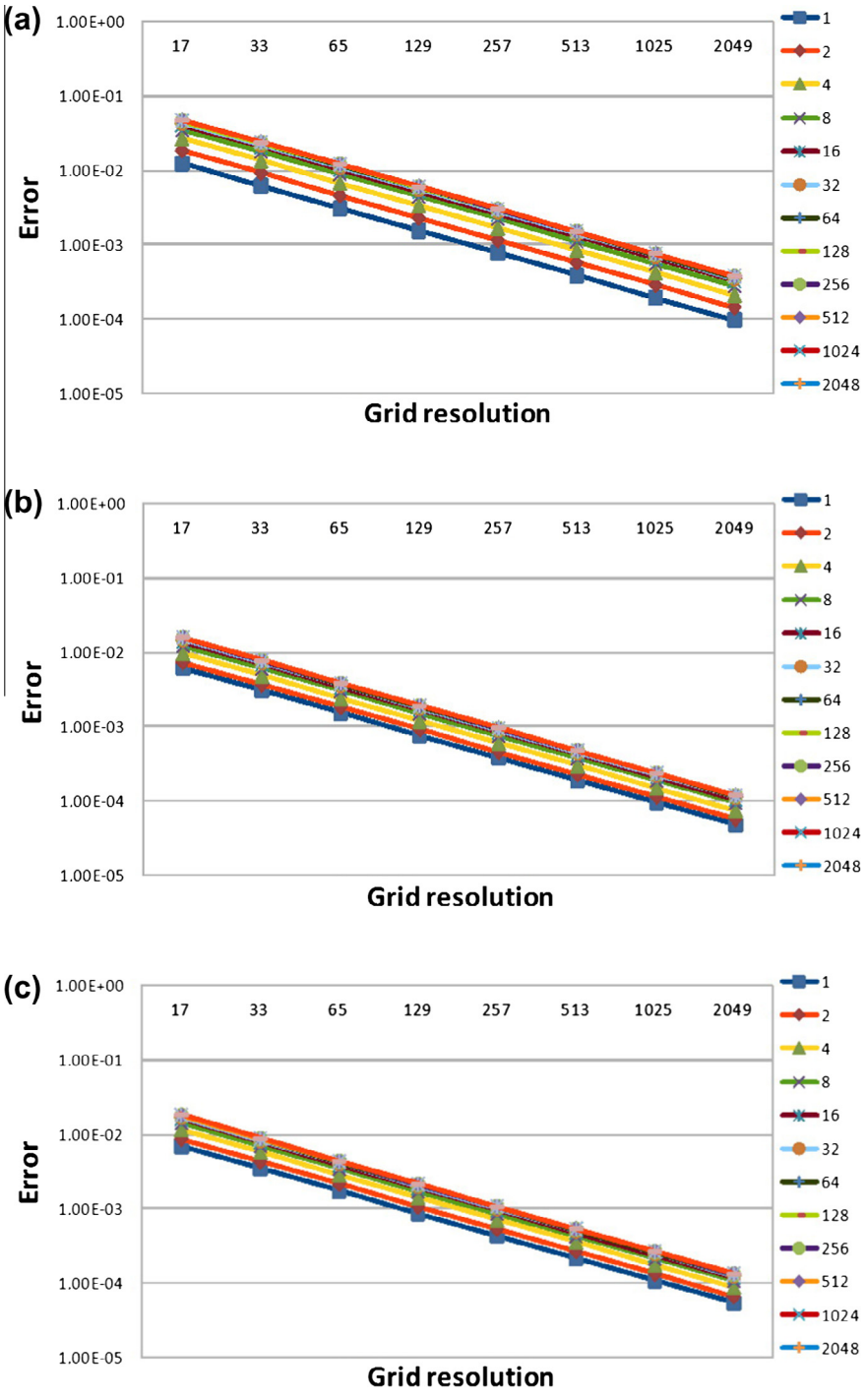

Fig. 1. (a) L_∞-error (b) L_2-error (c) L_1-error via grid resolution in logarithmic scale; different curves correspond to different viscosity contrasts (marked).

9. Numerical test

We compare the solution obtained using Woodbury formula with the explicitly known solution of the following configuration known as **solCx** problem. The analytic solution of the discussed test problem is given in [25]. The Stokes flow in a box $\Omega = [0,1] \times [0,1]$ with free slip boundary conditions is considered. There is a viscosity jump: the viscosity is equal to η_A in $x < \frac{1}{2}$ and is equal to η_B in $x \geqslant \frac{1}{2}$. The flow is driven by perturbation of density ρ, as follows:

$$\rho = -\sigma \sin(n_z \pi z) \cos(n_x \pi x). \tag{25}$$

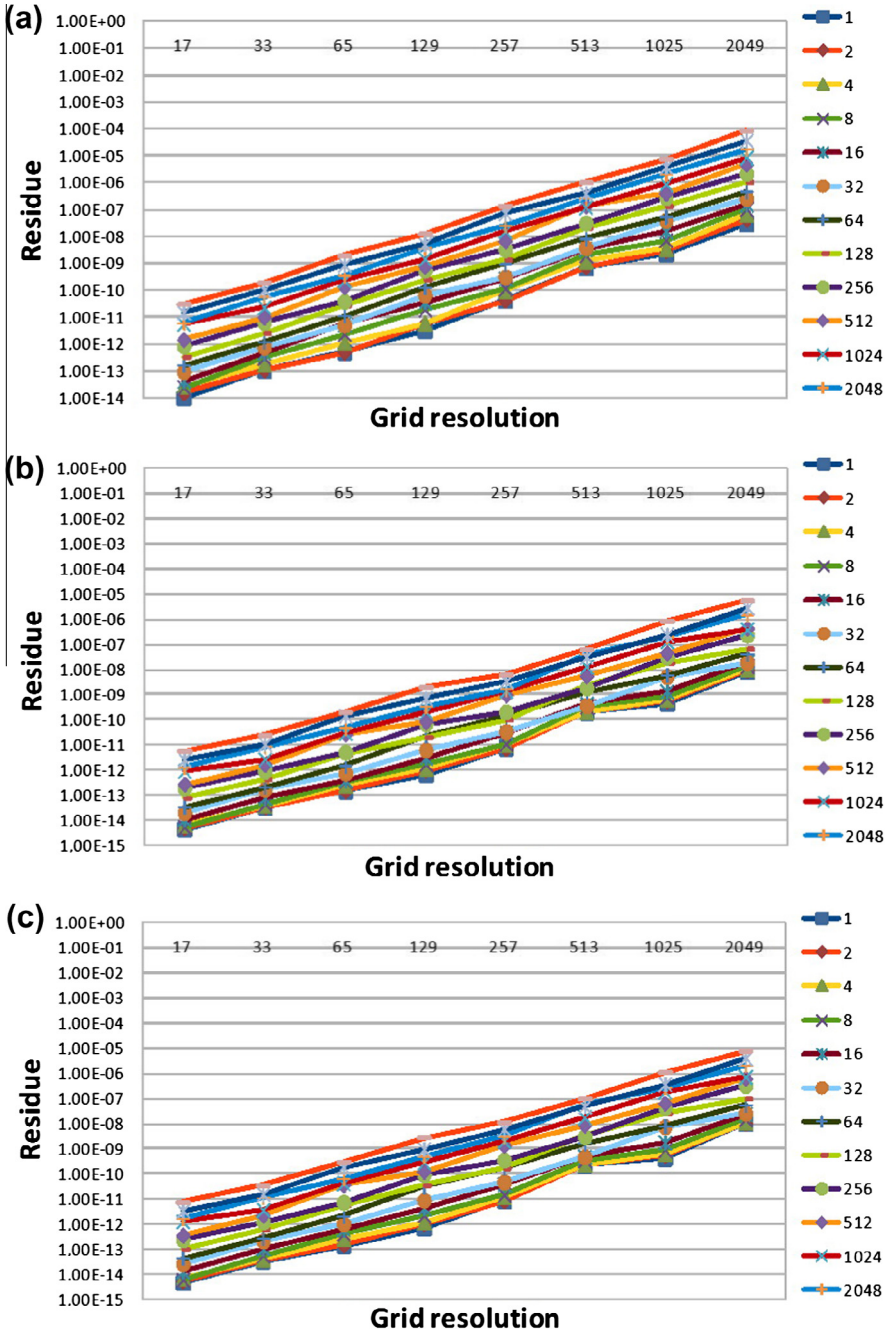

Fig. 2. (a) L_∞ (b) L_2 (c) L_1 norm of residue via grid resolution in logarithmic scale; different curves correspond to different viscosity contrasts (marked).

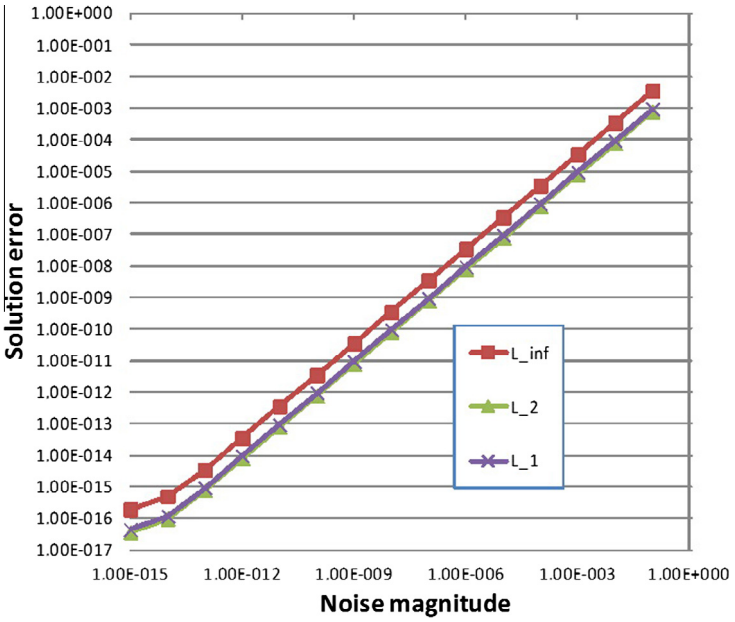

Fig. 3. Dependence of solution error on magnitude of noise added to body forces.

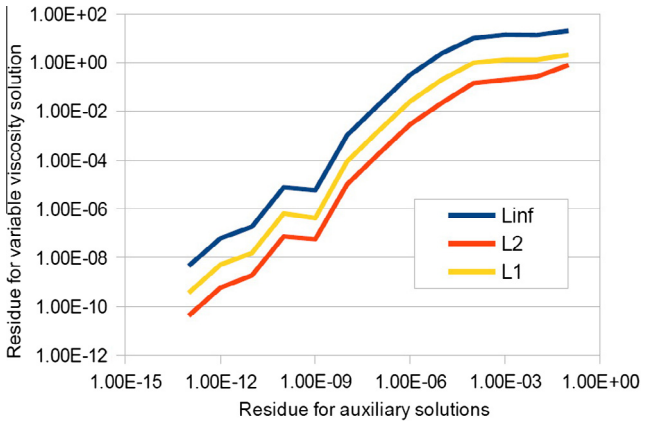

Fig. 4. Dependence of solution precision on precision of auxiliary problem with constant viscosity.

The variable parameters are chosen to be:

- density parameter: $\sigma = 1$.
- wave number in z domain: $n_z = 1$.
- wave number in x domain: $n_x = 2$.

Convergence of the algorithm is shown on Figs. 1 and 2. L_∞-error, L_2-error and L_1-error have the same order with a grid step.

Choice of relaxation parameters and solution time does not depend on the viscosities η_A and η_B, and this is one of the main advantages of the method based on Woodbury formula. However, in contrast to iterative methods the Woodbury formula has no error automatic correction property, hence precision of the solution has to be estimated. Numerical stability of the methods for perturbation of body forces is demonstrated on example for solCx problem in Fig. 3. Dependence of the solution precision on error of the solution of auxiliary unperturbed problem is shown in Fig. 4. One can see that the described method can be used to compute the solution with the error less than discretization error for viscosity contrast at least up to 10^6.

10. Conclusion

The suggested method has the following advantages:

1. The Woodbury formula provides non-iterative method to compute solution to Stokes and continuity equations in fixed time, which does not depend on viscosity contrast.
2. For very small rank of η_p, the Woodbury formula requires less computations than standard iterations techniques.

It should be noted that large number of operations in described approach is related with the inversion of the operator L_0 (for constant viscosity). Therefore for a fixed domain geometry this inversion should be made one time, and this result can be used later for different distributions of high viscosity inclusions. Moreover, for simple domain geometry one can use the discrete Fourier transform to find L_0^{-1}. The corresponding eigenfunctions for L_0 should be known, but if one studies, e.g., the flow in fixed domain for different distribution of inclusions (conventional situation), it is possible to find these eigenfunctions once (a preliminary operation) and then use this result for all these problems (the eigenfunctions depend only on the domain geometry and the boundary conditions only).

Thus, the following improvements to the described technique can be made:

1. For the case $\eta_r = 0$, the solution can be found much faster, using the Fourier transform to invert L_0.
2. A variant of multi-grid technique can be used for simultaneous calculation of z_k reducing complexity of the algorithm.

Acknowledgements

The work was made in the framework of Swiss–Russia Faculty Exchange Program. The work was partially financially supported by Government of Russian Federation (Grant 074-U01), by State contract of the Russian Ministry of Education and Science, by Grant of Russian Foundation for Basic Researches and Grants of the President of Russia (state contract 14.124.13.2045-MK and Grant MK-1493.2013.1).

References

[1] T. Gerya, Introduction to Numerical Geodynamic Modelling, Cambridge University Press, Cambridge, 2009.
[2] A. Ismail-Zadeh, P. Tackley, Computational Methods for Geodynamics, Cambridge University Press, Cambridge, 2010.
[3] T. Duretz, D.A. May, T.V. Gerya, P.J. Tackley, Discretization errors and free surface stabilization in the finite difference and marker-in-cell method for applied geodynamics: a numerical study, Geochem. Geophys. Geosyst. 12 (2011) Q07004, http://dx.doi.org/10.1029/2011GC003567.
[4] U. Christensen, Phase boundaries in finite amplitude mantle convection, Geophys. J. R. Astron. Soc. 68 (1982) 487–497.
[5] U. Christensen, An Eulerian technique for thermomechanical modeling of lithospheric extension, J. Geophys. Res. 97 (1992) 2015–2036.
[6] A. Lenardic, W.M. Kaula, A numerical treatment of geodynamic viscous flow problems involving the advection of material interfaces, J. Geophys. Res. 98 (1993) 8243–8260.
[7] B.M. Naimark, A.T. Ismail-Zadeh, Numerical models of subsidence mechanism in intracratonic basin: application to North American basins, Geophys. J. Int. 123 (1995) 149–160.
[8] B.M. Naimark, A.T. Ismail-Zadeh, An improved model of subsidence of heavy bodies in the asthenosphere, Comput. Seism. Geodyn. 3 (1996) 54–62.
[9] B.M. Naimark, A.T. Ismail-Zadeh, W.R. Jacoby, Numerical approach to problems of gravitational instability of geostructures with advected material boundaries, Geophys. J. Int. 134 (1998) 473–483.
[10] Y. Deubelbeiss, B.J.P. Kaus, J.A.D. Connolly, Direct numerical simulation of two-phase flow: effective rheology and flow patterns of particle suspensions, Earth Planet. Sci. Lett. 290 (12) (2010) 1–12.
[11] R. de, L. Kronig, W.G. Penney, Quantum mechanics of electrons in crystal lattices, Proc. R. Soc. London 130 A (1931) 499–513.
[12] S. Albeverio, F. Gesztesy, R. Hoegh-Krohn, H. Holden, Solvable Models in Quantum Mechanics with an appendix by P. Exner, second ed., AMS Chelsea Publishing, Providence, R.I., 2005.
[13] N. Bagraev, G. Martin, B.S. Pavlov, A. Yafyasov, Landau-Zener effect for a quasi-2D periodic sandwich, Nanosystems: Phys. Chem. Math. 2 (4) (2011) 32–50.
[14] I.S. Lobanov, V.Yu. Lotoreichik, I.Yu. Popov, Lower bound on the spectrum of the two-dimensional Schrödinger operator with a d-perturbation on a curve, Theor. Math. Phys. 162 (3) (2010) 332–340.
[15] I.Yu. Popov, Stokeslet and the operator extension theory, Revista Mat. Univ. Compl. Madrid 9 (1) (1996) 235–258.
[16] Yu.V. Gugel, I.Yu. Popov, S.L. Popova, Hydrotron: creep and slip, Fluid Dyn. Res. 18 (4) (1996) 199–210.
[17] I.Yu. Popov, Operator extensions theory and eddies in creeping flow, Phys. Scr. 47 (1993) 682–686.
[18] Y. Saad, Iterative Methods for Sparse Linear Systems, second ed., Society for Industrial and Applied Mathematics, Philadelphia, 2003.
[19] W.H. Press, S.A. Teukolsky, W.T. Vetterling, B.P. Flannery, Numerical recipes, third ed., The Art of Scientific Computing, Cambridge University Press, Cambridge, 2007.
[20] K.W. Morton, D.F. Mayers, Numerical Solution of Partial Differential Equations: An Introduction, second ed., Cambridge University Press, Camdridge, 2005.
[21] I.A. Shereshevskii, A finite dimension analog of the Krein Formula, J. Nonlinear Math. Phys. 8 (2001) 446–457.
[22] S. Chandrasekhar, Hydrodynamic and Hydromagnetic Stability, Dover Publications, New York, 1961.
[23] L.D. Landau, E.M. Lifschitz, Fluid Mechanics, Pergamon Press, New York, 1959.
[24] U. Trottenberg, C.W. Oosterlee, A. Schüler, Multi-grid, Academic Press, New York, 2001.
[25] S. Zhong, Analytic solutions for Stokes flow with lateral variations in viscosity, Geophys. J. Int. 124 (1996) 18–28.